Kruskal–Katona inequalities for toric g-vectors and interlacing of toric g-contribution polynomials

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Abstract

Strengthening a conjecture of Ehrenborg, Hetyei, and Readdy, we prove that the toric g-vector of a simple polytope satisfies the Kruskal–Katona inequalities whenever its γ-vector is nonnegative. More generally, if $n \in \mathbb{Z}_{\geq 0}$, $m = \lfloor n/2 \rfloor$, and $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_{\geq 0}^{m+1}$ with $\gamma_0 = 1$, then the coefficient vector of $\sum_{j=0}^{m} \gamma_j g_{n,j}(x)$ satisfies all Kruskal–Katona inequalities. We also prove that every row of the triangular array $(g_{n,j})$ is interlacing, giving an independent proof of a conjecture of Xiao also proved by Alexandersson. Consequently, every nonzero nonnegative linear combination of a row is real-rooted. In particular, the toric g-polynomial of every simple polytope with nonnegative γ-vector is real-rooted, including cyclohedra and chordal nestohedra. We further show that, for every integer $\ell \geq 0$, the sequence $\left(g_{\ell+2j,j}(x)\right)_{j\geq 0}$ is a generalized Sturm sequence.

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1 Introduction

Stanley introduced the generalized h-vector, now called the toric h-vector, and the associated toric g-vector [**?**]. For integers $n \geq 0$ and $0 \leq j \leq \lfloor n/2 \rfloor$, Ehrenborg, Hetyei, and Readdy [**?**] defined the toric g-contribution polynomials

$$g_{n,j}(x) = \sum_{k=0}^{\min\{\lfloor n/2 \rfloor, n-j\}} C_{n-j-k} \binom{n-k}{k} (x-1)^k, \qquad C_m = \frac{1}{m+1}\binom{2m}{m} \qquad (m \in \mathbb{Z}_{\geq 0}). \tag{1}$$

For background on the γ-expansion of a reciprocal h-polynomial, see Gal [**?**]. If P is an n-dimensional simple polytope whose ordinary h-polynomial has the expansion

$$h(P, x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j(P) x^j (1+x)^{n-2j},$$

then Ehrenborg, Hetyei, and Readdy [**?**] showed that

$$g(P, x) = \sum_{j=0}^{\lfloor n/2 \rfloor} \gamma_j(P) g_{n,j}(x). \tag{2}$$

For $r \geq 1$ and $A \in \mathbb{Z}_{>0}$, write the unique r-binomial expansion

$$A = \binom{p_r}{r} + \binom{p_{r-1}}{r-1} + \cdots + \binom{p_\ell}{\ell}, \qquad p_r > p_{r-1} > \cdots > p_\ell \geq \ell \geq 1,$$

where $p_r, \ldots, p_\ell$ are integers, and define

$$A^{\langle r\rangle} = \binom{p_r}{r+1} + \binom{p_{r-1}}{r} + \cdots + \binom{p_\ell}{\ell+1}, \qquad 0^{\langle r\rangle} = 0.$$

We index face numbers by cardinality. Thus v_r denotes the number of faces with r vertices, and $v_0 = 1$ counts the empty face. The Kruskal–Katona theorem [**?**, **?**] says that a finite nonnegative integral vector $(1, v_1, \ldots, v_m)$ is the face vector of a simplicial complex if and only if

$$v_{r+1} \leq v_r^{\langle r\rangle} \qquad (1 \leq r < m). \tag{3}$$

Throughout this paper, saying that a finite vector $(1, v_1, \ldots, v_m)$ satisfies the Kruskal–Katona inequalities means that its entries are nonnegative integers and that (**??**) holds. For the relation between γ-vectors and these inequalities, see Nevo and Petersen [**?**].

Ehrenborg, Hetyei, and Readdy [**?**, Conjecture 1.1] formulated the following conjecture.

Conjecture 1.1 (Ehrenborg–Hetyei–Readdy)**.** *If the γ-vector of a simple polytope satisfies the Kruskal–Katona inequalities, then so does its toric g-vector.*

Our first result shows that the Kruskal–Katona conditions on the γ-vector can be replaced by entrywise nonnegativity.

Theorem 1.2. *Fix an integer $n \geq 0$, put $m = \lfloor n/2 \rfloor$, and let*

$$\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_m) \in \mathbb{Z}_{\geq 0}^{m+1}, \qquad \gamma_0 = 1.$$

If

$$G(x) = \sum_{j=0}^{m} \gamma_j g_{n,j}(x) = \sum_{r=0}^{m} G_r x^r,$$

then $(G_0, G_1, \ldots, G_m)$ satisfies all Kruskal–Katona inequalities.

Corollary 1.3. *Conjecture **??** holds.*

Proof. Let P be a simple polytope whose γ-vector satisfies the Kruskal–Katona inequalities. Then $\gamma_0(P) = 1$ and every $\gamma_j(P)$ is a nonnegative integer. The conclusion follows from Theorem **??** and (**??**). □

We next turn to real-rootedness and interlacing. For background, see Brenti [**?**]. Ehrenborg, Hetyei, and Readdy [**?**, Conjectures 11.1 and 11.2] conjectured that the contribution polynomials and the toric g-polynomials of cyclohedra and chordal nestohedra are real-rooted. Xiao [**?**] later proved that every $g_{n,j}(x)$ is real-rooted and established the interlacings

$$g_{n,j}(x) \preceq g_{n+1,j}(x), \qquad g_{n,j}(x) \preceq g_{n+1,j+1}(x)$$

in their admissible ranges, where $\preceq$ denotes the interlacing relation defined in Section 3. At the suggestion of Athanasiadis, Xiao also formulated the conjecture that each row is interlacing [**?**, Conjecture 4.2]. We give an independent proof of this conjecture; see also Alexandersson [**?**, Theorem 8.1].

Theorem 1.4. *For every integer $n \geq 0$, the sequence*

$$g_{n,0}(x), g_{n,1}(x), \ldots, g_{n,\lfloor n/2 \rfloor}(x)$$

is interlacing.

In fact, the row theorem yields real-rootedness for every nonzero nonnegative linear combination of a row, not only for combinations arising from polytopes. In particular, we obtain the following result.

Theorem 1.5. *If P is a simple polytope with nonnegative γ-vector, then $g(P, x)$ is real-rooted.*

Postnikov, Reiner, and Williams [?] proved that cyclohedra and chordal nestohedra have nonnegative γ-vectors. Hence Theorem **??** applies to both families.

Xiao's results also cover the columns and the fixed-$n-j$ diagonals. Using the terminology of Liu and Wang [?], we show that the remaining natural diagonal family forms a generalized Sturm sequence as well.

Theorem 1.6. *For every integer $\ell \geq 0$, the sequence*

$$\big(g_{\ell+2j,j}(x)\big)_{j\geq 0}$$

is a generalized Sturm sequence.

The remainder of the paper is organized as follows. Section 2 proves Theorem **??**. Section 3 develops the combinatorial descriptions and proves row interlacing through moment identities, Chebyshev systems, and Jacobi polynomials. Section 4 establishes interlacing along the fixed-$n-2j$ diagonals.

2 Kruskal–Katona inequalities

Fix an integer $n \geq 0$, let γ and G be as in Theorem **??**, and write $n = 2m + \varepsilon$, where $\varepsilon \in \{0, 1\}$. The proof combines an upper bound for successive coefficients with a lower bound coming from the cube. For each integer $r \geq 1$, set $b_r(n, 0) = 0$ and define

$$b_r(n,i) = \frac{n+1-2i}{r}\binom{n-i}{r-1}\binom{i-1}{r-1} \qquad (1 \leq i \leq m). \tag{4}$$

Ehrenborg, Hetyei, and Readdy [?] showed that

$$[x^r]g_{n,j}(x) = \sum_{i=j}^{m}\left(\binom{n-2j}{i-j} - \binom{n-2j}{i-j-1}\right)b_r(n,i), \qquad 0 \leq j \leq m, \quad r \geq 1. \tag{5}$$

We use the convention $\binom{N}{k} = 0$ for $N \in \mathbb{Z}_{\geq 0}$ when $k < 0$ or $k > N$. The coefficients in parentheses are nonnegative because, for $j \leq i \leq m$,

$$\binom{n-2j}{i-j} - \binom{n-2j}{i-j-1} = \frac{n-2i+1}{n-i-j+1}\binom{n-2j}{i-j} \geq 0.$$

We also record the constant terms. Let $C(z) = \sum_{a\geq 0} C_a z^a$. Expanding $(z - z^2)^j C(z - z^2)$ and using (**??**) at $x = 0$ gives

$$g_{n,j}(0) = [z^n](z - z^2)^j C(z - z^2).$$

Since $C(z(1-z)) = (1-z)^{-1}$,

$$g_{n,j}(0) = [z^n]z^j(1-z)^{j-1} = \begin{cases} 1, & j = 0, \\ 0, & 1 \le j \le m. \end{cases} \tag{6}$$

In the second case, $z^j(1-z)^{j-1}$ has degree $2j-1 < n$.

Lemma 2.1. *For $1 \le r < m$, put*

$$\rho_{n,r} = \frac{(m-r)(m-r+1+\varepsilon)}{r(r+1)}.$$

Then

$$[x^{r+1}]g_{n,j}(x) \le \rho_{n,r}[x^r]g_{n,j}(x) \qquad (0 \le j \le m).$$

Consequently,

$$G_{r+1} \le \rho_{n,r} G_r. \tag{7}$$

Proof. If $i \le r$, then $b_{r+1}(n,i) = 0$. For $i \ge r+1$,

$$\frac{b_{r+1}(n,i)}{b_r(n,i)} = \frac{(i-r)(n-i-r+1)}{r(r+1)}. \tag{8}$$

For $r+1 \le i < m$, the forward difference of the numerator is $n - 2i > 0$. Hence the numerator attains its maximum over $r+1 \le i \le m$ at $i = m$, where it is $(m-r)(m-r+1+\varepsilon)$. Hence $b_{r+1}(n,i) \le \rho_{n,r} b_r(n,i)$. Applying this inequality termwise in (**??**) proves the coefficient estimate. Multiplying by $\gamma_j \ge 0$ and summing over j gives (**??**). $\square$

Since $\gamma_0 = 1$ and all γ_j are nonnegative,

$$G_r \ge [x^r]g_{n,0}(x). \tag{9}$$

The polynomial $g_{n,0}(x)$ is the toric g-polynomial of the n-dimensional cube. By [**?**, Corollary 4.9], $[x^r]g_{n,0}(x)$ is the number of noncrossing partitions of $[n]$ having exactly r nonsingleton blocks. For the classical theory of noncrossing partitions, see Kreweras [**?**].

Restricting to partitions whose nonsingleton blocks all have size two gives a useful lower bound. Choose the $2r$ elements in these blocks and then choose a noncrossing perfect matching in their induced linear order. Adding the remaining elements as singleton blocks preserves the noncrossing property. The chosen subset and matching are recovered uniquely from the resulting partition. There are C_r such matchings [**?**, Exercise 6.19(o)]. Therefore

$$[x^r]g_{n,0}(x) \ge C_r\binom{n}{2r}. \tag{10}$$

Lemma 2.2. *Assume $1 \le r < m$. Put $s = m - r$ and*

$$X = 2r + (r+1)\rho_{n,r} = 2r + \frac{s(s+1+\varepsilon)}{r}, \qquad q = \lceil X \rceil.$$

Then

$$C_r\binom{n}{2r} \ge \binom{q}{r}.$$

Proof. Since $n = 2r + 2s + \varepsilon$,

$$C_r \binom{n}{2r} = \frac{1}{(r+1)(r!)^2} \prod_{k=1}^{r} (2s + \varepsilon + k)(2s + \varepsilon + 2r + 1 - k). \tag{11}$$

We claim that, for every $1 \le k \le r$,

$$(2s + \varepsilon + k)(2s + \varepsilon + 2r + 1 - k) \ge (k+1)(X + 2 - k). \tag{12}$$

For $\varepsilon = 0$, r times the difference between the two sides of (**??**) is

$$(4r - k - 1)s^2 + (4r^2 + 2r - k - 1)s - 2r(r+1).$$

This expression is increasing for $s \ge 1$, and at $s = 1$ it equals $2(r^2 + 2r - k - 1) \ge 2(r^2 + r - 1) > 0$. For $\varepsilon = 1$, r times the difference between the two sides of (**??**) exceeds its value for $\varepsilon = 0$ by

$$2r^2 + 4rs + 2r - (k+1)s \ge 2r^2 + 2rs + 2r + s(r-1) > 0.$$

Thus (**??**) holds for both parities of n. Since $X \ge 2r$ and $1 \le k \le r$, all factors on both sides of (**??**) are positive. Multiplying these inequalities for $k = 1, \ldots, r$ and using (**??**), we obtain

$$\begin{aligned} C_r \binom{n}{2r} &\ge \frac{\prod_{k=1}^{r} (k+1)(X + 2 - k)}{(r+1)(r!)^2} \\ &= \frac{1}{r!} \prod_{k=1}^{r} (X + 2 - k). \end{aligned}$$

Since $q = \lceil X \rceil \le X + 1$, each factor satisfies $X + 2 - k \ge q + 1 - k$. Therefore

$$C_r \binom{n}{2r} \ge \frac{1}{r!} \prod_{k=1}^{r} (q + 1 - k) = \binom{q}{r},$$

as desired. □

The final ingredient is the following elementary estimate.

Lemma 2.3. *Let $A \in \mathbb{Z}_{\ge 0}$, $r \in \mathbb{Z}_{\ge 1}$, $\theta \in \mathbb{R}_{\ge 0}$, and $q \in \mathbb{Z}$ satisfy $q \ge 2r + (r+1)\theta$. If $A \ge \binom{q}{r}$, then $A^{\langle r \rangle} \ge \theta A$.*

Proof. The hypothesis on q gives $q \ge 2r$, so $\binom{q}{r} > 0$ and hence $A > 0$. Thus A has the binomial representation introduced above. Let p be its leading upper index, so that

$$\binom{p}{r} \le A < \binom{p+1}{r}.$$

The assumption $A \ge \binom{q}{r}$ implies $p \ge q$, while the definition of $A^{\langle r \rangle}$ gives

$$A^{\langle r \rangle} \ge \binom{p}{r+1}.$$

Consequently,

$$\frac{A^{\langle r \rangle}}{A} > \frac{\binom{p}{r+1}}{\binom{p+1}{r}}$$

$$
\begin{aligned}
&= \frac{(p-r)(p-r+1)}{(r+1)(p+1)}\\
&= \frac{p-2r}{r+1} + \frac{r}{p+1}\\
&\ge \frac{q-2r}{r+1} \ge \theta.
\end{aligned}
$$

$\square$

Proof of Theorem **??**. Equation (**??**) shows that every G_r is an integer, while (**??**) gives $G_0 = 1$. For $r \ge 1$, (**??**) and $\gamma_j \ge 0$ give $G_r \ge 0$. If $m \le 1$, there are no inequalities in (**??**) to verify. Assume $m \ge 2$ and fix $1 \le r < m$.

Let q be as in Lemma **??**. Equations (**??**) and (**??**), together with that lemma, give $G_r \ge [x^r]g_{n,0}(x) \ge C_r\binom{n}{2r} \ge \binom{q}{r}$. Moreover, $q \ge X = 2r + (r+1)\rho_{n,r}$. Lemma **??**, with $A = G_r$ and $\theta = \rho_{n,r}$, gives $G_r^{\langle r\rangle} \ge \rho_{n,r}G_r$, while Lemma **??** gives $G_{r+1} \le \rho_{n,r}G_r$. Hence $G_{r+1} \le G_r^{\langle r\rangle}$ for every $1 \le r < m$, as required. $\square$

3 Row interlacing and real-rootedness

We first fix our interlacing conventions. Let $f, g \in \mathbb{R}[x]$ be real-rooted polynomials with positive leading coefficients. If both have degree d, with zeros $a_1 \le \cdots \le a_d$ and $b_1 \le \cdots \le b_d$, respectively, then $f \preceq g$ means that

$$a_1 \le b_1 \le a_2 \le b_2 \le \cdots \le a_d \le b_d.$$

If $\deg g = \deg f + 1 = d + 1$, then $f \preceq g$ means that

$$b_1 \le a_1 \le b_2 \le \cdots \le a_d \le b_{d+1}.$$

We write $f \prec g$ when all displayed inequalities are strict. If f and g are nonzero constants, we regard $f \preceq g$ as vacuously true. The same convention applies when f is a nonzero constant and g is a real-rooted linear polynomial with positive leading coefficient. Following Liu and Yan [**?**], we call a finite sequence *interlacing* if $f_i \preceq f_j$ whenever $i < j$. We use the term *generalized Sturm sequence*, as in Liu and Wang [**?**], for a finite or infinite sequence whose consecutive terms interlace.

Example (the row $n = 4$). Equation (**??**) gives

$$g_{4,0}(x) = 2x^2 + 11x + 1, \qquad g_{4,1}(x) = x^2 + 4x, \qquad g_{4,2}(x) = x^2 + x.$$

If $a_\pm = (-11 \pm \sqrt{113})/4$ are the zeros of $g_{4,0}$, then

$$a_- < -4 < a_+ < 0, \qquad a_- < -1 < a_+ < 0, \qquad -4 < -1 < 0 = 0.$$

Thus the row is interlacing. The common zero 0 explains the use of weak interlacing.

3.1 Combinatorial descriptions

Let $\mathcal{D}_n$ be the set of Dyck words of semilength n. For $w \in \mathcal{D}_n$, let $\mathrm{uud}(w)$ and $\mathrm{udd}(w)$ denote the numbers of factors UUD and UDD, respectively.

Proposition 3.1. *For integers $n \ge 0$ and $0 \le j \le \lfloor n/2 \rfloor$,*

$$\binom{n-j}{j} g_{n,j}(x) = \sum_{w \in \mathcal{D}_n} \binom{\mathrm{udd}(w)}{j} x^{\mathrm{uud}(w)}. \tag{13}$$

Proof. For $n = 0$, we have $j = 0$, and both sides are 1. Assume $n \geq 1$. Following Ehrenborg, Hetyei, and Readdy, label the down-steps of $w \in \mathcal{D}_n$ from left to right by $D_1, \ldots, D_n$, and label a factor UD_bD_{b+1} by b. Count pairs (w, B) with weight $x^{\mathrm{uud}(w)}$, where B is a j-element set of such labels. For a fixed word w, no two such labels are consecutive, and there are $\mathrm{udd}(w)$ of them. Counting first by w gives

$$\sum_{w \in \mathcal{D}_n} \binom{\mathrm{udd}(w)}{j} x^{\mathrm{uud}(w)}.$$

Conversely, the possible label sets are the j-element subsets of $[n-1]$ with no consecutive elements, of which there are $\binom{n-j}{j}$. For each fixed B, [**?**, Proposition 4.5] shows that the weighted contribution of the $(\varnothing, B)$-compatible Dyck words is $g_{n,j}(x)$. Counting first by B gives $\binom{n-j}{j} g_{n,j}(x)$, proving the identity. □

We also use the following specialization of the compatibility condition from [**?**, Section 4]. If the down-steps are labeled from left to right by $D_1, \ldots, D_n$ and $B \subseteq [n-1]$ has no consecutive elements, then a Dyck word is $(\varnothing, B)$-compatible if and only if UD_bD_{b+1} is a factor for every $b \in B$.

A path compression of the compatibility model gives a second combinatorial description. By a Łukasiewicz path we mean a path that starts at height 0, has increments in $\{-1, 0, 1, \ldots\}$, stays nonnegative, and ends at height 0. Fix integers $n \geq 0$ and $0 \leq j \leq \lfloor n/2 \rfloor$. Put $B_0 = \{1, 3, \ldots, 2j-1\}$, where $B_0 = \varnothing$ when $j = 0$, and let $\mathcal{D}_n(B_0)$ be the set of $(\varnothing, B_0)$-compatible Dyck words of semilength n. By [**?**, Proposition 4.5],

$$g_{n,j}(x) = \sum_{w \in \mathcal{D}_n(B_0)} x^{\mathrm{uud}(w)}.$$

For $w \in \mathcal{D}_n(B_0)$, write $w = U^{\nu_1} D_1 \cdots U^{\nu_n} D_n$. The increments $\nu_i - 1$ form a Łukasiewicz path. Since $UD_{2r-1}D_{2r}$ is a factor for every $1 \leq r \leq j$, we have

$$(\nu_{2r-1}, \nu_{2r}) = (h_r, 0), \qquad h_r \geq 1.$$

For each r, compress the two increments $h_r - 1$ and -1 into the single increment $h_r - 2$. Here $h_r = 1$ gives a down-step, $h_r = 2$ a level step, and $h_r \geq 3$ a positive step. The marked block contains a factor UUD exactly when $h_r \geq 2$. The corresponding compressed step therefore has weight 1 when it is a down-step and weight x otherwise. The uncompressed increments have indices $2j+1, \ldots, n$. At such an index i, a factor UUD occurs exactly when $\nu_i \geq 2$, equivalently when the corresponding increment is positive. The remaining down-steps and level steps therefore have weight 1, while the positive steps have weight x.

The compressed path has length $n - j$, and the heights at all surviving vertices are unchanged. Conversely, expand each of the first j compressed increments $s \geq -1$ into the pair $(s+1, -1)$. If the height before this step is H, the intermediate and final heights are $H + s + 1$ and $H + s$, so nonnegativity is preserved in both directions. In the expanded path, this pair has the form $(h_r - 1, -1)$ with $h_r = s + 2 \geq 1$, and hence restores the factor $UD_{2r-1}D_{2r}$. Thus expansion is inverse to compression, and $g_{n,j}$ enumerates the compressed paths with the stated weights.

The defining formula also gives the following generating-function identity. Put $y = z + (x-1)z^2$. Expanding $C(y)$ gives

$$\begin{aligned}
[z^n] y^j C(y) &= [z^n] \sum_{a \geq 0} C_a z^{a+j} \big(1 + (x-1)z\big)^{a+j} \\
&= \sum_{k=0}^{\min\{\lfloor n/2 \rfloor, n-j\}} C_{n-j-k} \binom{n-k}{k} (x-1)^k = g_{n,j}(x).
\end{aligned}$$

Hence

$$g_{n,j}(x) = [z^n]\big(z+(x-1)z^2\big)^j C\big(z+(x-1)z^2\big). \tag{14}$$

Factoring one copy of $z+(x-1)z^2$ gives the recurrence [**?**, Lemma 5.1]

$$g_{n,j}(x) = g_{n-1,j-1}(x) + (x-1)g_{n-2,j-1}(x) \qquad (j \geq 1,\ n \geq 2j). \tag{15}$$

3.2 Moment identities

We next derive vanishing moment conditions that locate the zeros of $R_1, \ldots, R_d$, defined below, in $(0,1)$. Let $n \geq 2$ and write $n = 2d + \varepsilon$, where $\varepsilon \in \{0,1\}$, and put $\beta = \varepsilon - \frac{1}{2}$. Set

$$F_j(u) = u^d g_{n,j}(1-u^{-1}) \quad (0 \leq j \leq d), \qquad R_j(u) = \frac{F_j(u)}{u-1} \quad (1 \leq j \leq d). \tag{16}$$

Equation (**??**) gives

$$F_j(u) = \sum_{k=0}^{d} (-1)^k C_{n-j-k} \binom{n-k}{k} u^{d-k}.$$

In particular, $[u^d]F_j = C_{n-j} > 0$. For $j \geq 1$, we have $F_j(1) = 0$, so R_j has degree $d-1$ and positive leading coefficient C_{n-j}. For $c \in \mathbb{R}$ and $p \in \mathbb{R}[u]$, define $(\mathcal{E}_c p)(u) = up'(u) + cp(u)$. The Catalan recurrence gives

$$4\mathcal{E}_{d-j+\beta}F_{j+1} = \mathcal{E}_{d-j+\beta+3/2}F_j \qquad (0 \leq j < d). \tag{17}$$

For $0 \leq k \leq d$, equality of the coefficients of $(-1)^k u^{d-k}\binom{n-k}{k}$ reduces to

$$4\big(n-j-k-\tfrac{1}{2}\big)C_{n-j-k-1} = (n-j-k+1)C_{n-j-k},$$

which follows from $C_{r+1}/C_r = 2(2r+1)/(r+2)$ upon setting $r = n-j-k-1$.

Lemma 3.2. *Let $p \in \mathbb{R}[u]$ have positive leading coefficient and only positive zeros. If $0 < a < b$, then*

$$\mathcal{E}_a p \preceq \mathcal{E}_b p \preceq p.$$

Moreover, $\mathcal{E}_c p$ has only positive zeros for every $c > 0$. The interlacing is strict except at zeros inherited from multiple zeros of p.

Proof. If $\deg p = 0$, then p is a positive constant and $\mathcal{E}_c p = cp$, so the assertions are immediate. Hence assume $D = \deg p \geq 1$. First suppose that p has simple zeros $0 < r_1 < \cdots < r_D$. Away from these zeros, set

$$\psi(u) = \frac{up'(u)}{p(u)} = D + \sum_{i=1}^{D} \frac{r_i}{u-r_i}.$$

Then $\psi'(u) = -\sum_{i=1}^{D} \frac{r_i}{(u-r_i)^2} < 0$. For $u < 0$,

$$\psi(u) = \sum_{i=1}^{D} \frac{u}{u-r_i} > 0.$$

For $c > 0$, the equation $(\mathcal{E}_c p)(u) = 0$ is equivalent to $\psi(u) = -c$. There is no negative solution. There is one solution in $(0, r_1)$, one in each interval (r_i, r_{i+1}), and none to the right of r_D. Since

$\deg \mathcal{E}_c p = D$, these are all the zeros of $\mathcal{E}_c p$. Each zero moves strictly to the right as c increases. Thus, for $0 < a < b$,

$$\mathcal{E}_a p \prec \mathcal{E}_b p \prec p.$$

When p has multiple zeros, perturb its zeros to nearby distinct positive numbers and pass to the limit. Since $(\mathcal{E}_c p)(0) = cp(0) \neq 0$, the limiting zeros remain positive. Interlacing is closed under coefficientwise limits. It remains to determine when equality can occur. If $r > 0$ is a zero of multiplicity m and $p(u) = (u-r)^m q(u)$ with $q(r) \neq 0$, then

$$(\mathcal{E}_c p)(u) = (u-r)^{m-1}\big(muq(u) + (u-r)(uq'(u) + cq(u))\big).$$

The factor in parentheses takes the nonzero value $mrq(r)$ at $u = r$. Thus r has multiplicity exactly $m-1$ in every $\mathcal{E}_c p$. Moreover, a common zero of $\mathcal{E}_a p$ and $\mathcal{E}_b p$ must be a zero of p, since their difference is $(b-a)p$. Hence equality occurs only at zeros inherited from multiple zeros of p. □

We combine these moment conditions with a Chebyshev-system argument. We use standard facts about Chebyshev systems from Karlin and Studden [**?**, Chapter I, §4]. An ordered family $\varphi_1, \dots, \varphi_N$ of continuous real functions on an interval I is a *Chebyshev system* (or *T-system*) if every nonzero linear combination of the φ_i has at most $N-1$ distinct zeros in I. Equivalently, the determinants

$$\det\big[\varphi_p(u_q)\big]_{p,q=1}^{N}$$

are nonzero and have a fixed sign whenever $u_1 < \cdots < u_N$ in I.

For the monomials needed below, this criterion follows directly from the bialternant formula for Schur polynomials [**?**, Theorem 7.15.1]. Let $0 \le \lambda_1 < \cdots < \lambda_N$ be nonnegative half-integers and $0 < u_1 < \cdots < u_N$. Put $m_p = 2\lambda_p$ and $y_q = \sqrt{u_q}$, and define $\mu = (\mu_1, \dots, \mu_N)$ by $\mu_i = m_{N+1-i} - (N-i)$ for $1 \le i \le N$. Since the integers $m_1 < \cdots < m_N$ are strictly increasing,

$$\mu_i - \mu_{i+1} = m_{N+1-i} - m_{N-i} - 1 \ge 0 \quad (1 \le i < N), \qquad \mu_N = m_1 \ge 0.$$

Thus μ is a partition, and the bialternant formula gives

$$\det\big[u_q^{\lambda_p}\big]_{p,q=1}^{N} = \det\big[y_q^{m_p}\big]_{p,q=1}^{N} = \prod_{1 \le p < q \le N} (y_q - y_p)\, s_\mu(y_1, \dots, y_N) > 0.$$

Indeed, a Schur polynomial has nonnegative monomial coefficients, so $s_\mu(y_1, \dots, y_N) > 0$ when all y_q are positive. Therefore $u^{\lambda_1}, \dots, u^{\lambda_N}$ form a Chebyshev system on $(0,1)$.

We use the following standard sign-change consequence [**?**, Chapter I, pp. 20–21].

Lemma 3.3. *Let $\varphi_1, \dots, \varphi_N$ be a Chebyshev system on an interval I, and let ω be positive on I. Suppose that the displayed integrals converge. If a nonzero continuous function H satisfies*

$$\int_I H(u)\varphi_i(u)\omega(u)\,du = 0 \qquad (1 \le i \le N),$$

then H has at least N sign changes in I. In particular, if H is a polynomial of degree at most N, then it has degree N and all its zeros are simple and lie in I.

For $1 \le j \le d$, define the exponent set

$$\Lambda_j = \{0, \dots, d-j-1\} \cup \{d-j+\tfrac{3}{2}, d-j+\tfrac{5}{2}, \dots, d-\tfrac{1}{2}\}, \tag{18}$$

where either displayed set is understood to be empty if its lower endpoint exceeds its upper endpoint.

Lemma 3.4. *The set Λ_j has $d-1$ elements, and*

$$\int_0^1 R_j(u)u^{\beta+\lambda}(1-u)\,du = 0 \qquad (\lambda \in \Lambda_j). \tag{19}$$

Proof. The two blocks of Λ_j have $d-j$ and $j-1$ elements, so $|\Lambda_j| = d-1$. The moment assertion is vacuous when $d=1$. Assume $d \geq 2$. For $a \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$, we use the rising factorial $(a)_0 = 1$ and $(a)_m = a(a+1)\cdots(a+m-1)$ for $m \geq 1$. We begin with the following identity of rational functions in ℓ:

$$\sum_{k=0}^{d} \frac{(-1)^k C_{n-1-k}\binom{n-k}{k}}{d-k+\beta+\ell+1} = -\frac{2^{3d-1+\varepsilon}\prod_{r=0}^{d-2}(\ell-r)}{\prod_{q=0}^{d}(2\ell+2q+2\varepsilon+1)}. \tag{20}$$

The only possible finite poles of either side are $\ell_q = -q-\varepsilon-\frac{1}{2}$ $(0 \leq q \leq d)$. These points are pairwise distinct, and each is at most a simple pole on both sides. At ℓ_q, only the term $k = d-q$ on the left has a pole, and its residue is

$$(-1)^{d-q} C_{d+\varepsilon+q-1}\binom{d+\varepsilon+q}{d-q} = 2^{2d-2+\varepsilon}(-1)^{d-q}\frac{(q+\varepsilon+\frac{1}{2})_{d-1}}{q!(d-q)!}.$$

The equality follows from the Catalan formula. On the right, the vanishing factor has derivative 2, while

$$\prod_{\substack{0 \leq p \leq d \\ p \neq q}} (2\ell_q + 2p + 2\varepsilon + 1) = 2^d(-1)^q q!(d-q)!, \qquad \prod_{r=0}^{d-2}(\ell_q - r) = (-1)^{d-1}(q+\varepsilon+\tfrac{1}{2})_{d-1}.$$

Thus the residue on the right is the same. The difference has no finite poles and tends to 0 at infinity. This proves (**??**). For $0 \leq \ell \leq d-2$, its right-hand side vanishes. Since

$$\int_0^1 F_1(u)u^{\beta+\ell}\,du = \sum_{k=0}^{d} \frac{(-1)^k C_{n-1-k}\binom{n-k}{k}}{d-k+\beta+\ell+1},$$

it follows that

$$\int_0^1 F_1(u)u^{\beta+\ell}\,du = 0 \qquad (0 \leq \ell \leq d-2).$$

For $1 \leq j \leq d$ and $\lambda \geq 0$, set

$$M_j(\lambda) = \int_0^1 R_j(u)u^{\beta+\lambda}(1-u)\,du = -\int_0^1 F_j(u)u^{\beta+\lambda}\,du.$$

Thus $M_1(\ell) = 0$ for $0 \leq \ell \leq d-2$. If $H \in \mathbb{R}[u]$, $c \in \mathbb{R}$, and $\tau > -1$, then $u^{\tau+1}H(u) \to 0$ as $u \to 0^+$, and integration by parts gives

$$\int_0^1 (\mathcal{E}_c H)(u)u^\tau\,du = H(1) + (c-\tau-1)\int_0^1 H(u)u^\tau\,du. \tag{21}$$

All exponents used below satisfy $\tau \geq -\frac{1}{2}$. Applying this identity to (**??**) gives

$$4(\lambda-d+j+1)M_{j+1}(\lambda) = (\lambda-d+j-\tfrac{1}{2})M_j(\lambda) \qquad (1 \leq j < d). \tag{22}$$

Assume the assertion holds for M_j, where $1 \le j < d$. Since

$$\Lambda_{j+1} = \big(\Lambda_j \setminus \{d-j-1\}\big) \cup \{d-j+\tfrac{1}{2}\},$$

if $\lambda \in \Lambda_j \setminus \{d-j-1\}$, then $\lambda - d + j + 1 \neq 0$, and (**??**) gives $M_{j+1}(\lambda) = 0$. At the new exponent $\lambda = d-j+\frac{1}{2}$, the right-hand factor in (**??**) vanishes while the left-hand factor equals 6. Thus $M_{j+1}(d-j+\frac{1}{2}) = 0$, completing the induction. □

Proposition 3.5. *For $1 \le j < d$, the polynomials R_j and R_{j+1} have simple zeros in $(0,1)$, satisfy $R_j \preceq R_{j+1}$, and hence satisfy $F_j \preceq F_{j+1}$.*

Proof. Apply Lemma **??** on $(0,1)$ to the Chebyshev system $\{u^\lambda : \lambda \in \Lambda_j\}$ with weight $\omega(u) = u^\beta(1-u)$. Lemma **??** then shows that each R_j has degree $d-1$ and $d-1$ simple zeros in $(0,1)$. The common moments also control every real linear combination of adjacent polynomials. Indeed, Λ_j and Λ_{j+1} have $d-2$ common exponents. Let $H = AR_j + BR_{j+1}$ be a nonzero real linear combination. When $d = 2$, H has degree at most one and is therefore real-rooted. Assume $d \ge 3$. The common moment conditions show that H has at least $d-2$ distinct zeros in $(0,1)$. If its leading coefficient cancels, then $\deg H \le d-2$. Lemma **??** then gives $\deg H = d-2$, so these are all its zeros. Otherwise $\deg H = d-1$. After the $d-2$ real linear factors are removed, the remaining quotient has degree one. Thus every nonzero real linear combination of R_j and R_{j+1} is real-rooted. Both polynomials have degree $d-1$ and positive leading coefficient. The equal-degree form of the Hermite–Kakeya–Obreschkoff theorem [**?**, **?**, **?**] therefore shows that their zeros interlace in one of the two possible orientations.

It remains to determine the orientation. Let S_j be the sum of the zeros of R_j. The coefficient of u^{d-1} in F_j gives

$$S_j = (n-1)\frac{C_{n-j-1}}{C_{n-j}} - 1 = (n-1)\frac{n-j+1}{2(2n-2j-1)} - 1.$$

A direct calculation gives

$$S_{j+1} - S_j = \frac{3(n-1)}{2(2n-2j-3)(2n-2j-1)} > 0.$$

The reverse orientation would give $S_{j+1} \le S_j$, so $R_j \preceq R_{j+1}$. Multiplying both polynomials by $u-1$ adjoins the common largest zero 1. Hence $F_j \preceq F_{j+1}$. □

The preceding moment argument applies only to $F_1, \dots, F_d$. The boundary pair F_0, F_1 therefore requires a separate argument. Put

$$\Theta(u) = 4F_1(u) - F_0(u). \tag{23}$$

The case $j = 0$ of (**??**) is equivalent to

$$F_0 = \tfrac{2}{3}\mathcal{E}_{d+\beta}\Theta, \qquad F_1 = \tfrac{1}{6}\mathcal{E}_{d+\beta+3/2}\Theta. \tag{24}$$

Lemma 3.6. *The polynomial Θ has d simple positive zeros, of which $d-1$ lie in $(0,1)$ and one lies in $(1,\infty)$.*

Proof. For $d = 1$, direct calculation gives $\Theta(u) = 2u - 3$ when $n = 2$ and $\Theta(u) = 3u - 4$ when $n = 3$. Assume $d \ge 2$. The initial moments give, for $0 \le \ell \le d-2$,

$$\int_0^1 F_1(u) u^{\beta+\ell}\, du = 0.$$

Moreover, $F_0(1) = g_{n,0}(0) = 1$ and $F_1(1) = g_{n,1}(0) = 0$. Applying (**??**) to the case $j = 0$ of (**??**) therefore yields

$$0 = 1 + \big(d - \ell + \tfrac{1}{2}\big) \int_0^1 F_0(u) u^{\beta+\ell}\, du.$$

Since $\Theta(u) = 4F_1(u) - F_0(u)$, we obtain

$$\int_0^1 \Theta(u) u^{\beta+\ell}\, du = \frac{1}{d - \ell + 1/2} \qquad (0 \le \ell \le d-2). \tag{25}$$

For $0 \le r \le d-2$, expanding $(u-1)^r$ and using (**??**) gives

$$\int_0^1 \Theta(u)(u-1)^r u^\beta\, du = \sum_{\ell=0}^{r} (-1)^{r-\ell} \binom{r}{\ell} \frac{1}{d - \ell + 1/2} = \frac{r!}{\prod_{q=0}^{r}(d - r + 1/2 + q)}.$$

It follows that if $h(u) = \sum_{r=0}^{d-2} \widehat{h}_r (u-1)^r$ has degree at most $d-2$, then

$$\int_0^1 \Theta(u) h(u) u^\beta\, du = \sum_{r=0}^{d-2} \widehat{h}_r \frac{r!}{\prod_{q=0}^{r}(d - r + 1/2 + q)}. \tag{26}$$

Suppose, to the contrary, that Θ has at most $d-2$ sign changes in $(0,1)$, at $t_1 < \cdots < t_s$, and set $h(u) = -\prod_{i=1}^{s}(u - t_i)$. Since $\Theta(1) = 4F_1(1) - F_0(1) = -1$ and $h(1) < 0$, their product is positive just to the left of 1. On crossing any t_i, both Θ and h change sign. Zeros of Θ that are not sign changes do not alter its sign. Consequently, Θh is nonnegative on $(0,1)$ and positive on a nonempty open subinterval, so the left side of (**??**) is positive. But every coefficient of $h(1+y) = -\prod_{i=1}^{s}(y + 1 - t_i)$ is negative, so the right side is negative, a contradiction. Thus Θ has at least $d-1$ sign changes in $(0,1)$. Finally,

$$[u^d]\Theta = 4C_{n-1} - C_n = \frac{6C_{n-1}}{n+1} > 0.$$

Since $\Theta(1) = -1$, there is one further zero in $(1,\infty)$. Since $\deg \Theta = d$, these are all its zeros, and all are simple. $\square$

Lemma 3.7. *The polynomial F_0 has d simple positive zeros $s_1 < \cdots < s_d$. If $r_1 < \cdots < r_{d-1}$ are the zeros of F_d other than 1, then*

$$s_i < r_i < s_{i+1} \quad (1 \le i < d), \qquad s_d < 1.$$

Hence $F_0 \prec F_d$.

Proof. Since $d \ge 1$ and $\beta \in \{-\frac{1}{2}, \frac{1}{2}\}$,

$$0 < d + \beta < d + \beta + \tfrac{3}{2}.$$

Equations (**??**), together with Lemmas **??** and **??**, give $F_0 \prec F_1$. Since $F_1 = (u-1)R_1$ and all zeros of R_1 lie in $(0,1)$, the zero 1 is the largest zero of F_1. If $d = 1$, this already proves the assertion. Hence assume $d \ge 2$.

For $a \in \mathbb{R}$, put

$$\binom{a}{0} = 1, \qquad \binom{a}{q} = \frac{a(a-1)\cdots(a-q+1)}{q!} \quad (q \in \mathbb{Z}_{\ge 1}).$$

For $m \in \mathbb{Z}_{\geq 0}$ and $\alpha, \delta > -1$, we use the standard normalization [**?**, Eq. (4.3.2)]

$$P_m^{(\alpha,\delta)}(2u-1) = \sum_{q=0}^{m} \binom{m+\alpha}{m-q} \binom{m+\delta}{q} (u-1)^q u^{m-q},$$

and verify by coefficient comparison that

$$\mathcal{E}_{d+\beta+3/2} F_0 = 2^n P_d^{(0,\beta)}(2u-1), \qquad \frac{F_d}{u-1} = \frac{2^\varepsilon}{d} P_{d-1}^{(1,\beta+3/2)}(2u-1). \tag{27}$$

For $0 \leq k \leq d$, the coefficient of u^{d-k} in $\mathcal{E}_{d+\beta+3/2} F_0$ is

$$(-1)^k (n+1-k) C_{n-k} \binom{n-k}{k}.$$

On the right-hand side, Vandermonde's identity yields

$$\begin{aligned} 2^n (-1)^k \sum_{q=k}^{d} \binom{d}{q} \binom{d+\beta}{q} \binom{q}{k} \\ = 2^n (-1)^k \binom{d}{k} \binom{2d+\beta-k}{d}. \end{aligned}$$

The Catalan formula gives the first equality below, and the duplication formula for the gamma function gives the second:

$$(n+1-k) C_{n-k} \binom{n-k}{k} = \frac{(2n-2k)!}{k!(n-k)!(n-2k)!} = 2^n \binom{d}{k} \binom{2d+\beta-k}{d}.$$

For the second identity, multiply the asserted equality by $u-1$. The coefficient of u^{d-k} on its right-hand side is

$$\begin{aligned} \frac{2^\varepsilon}{d} (-1)^k \sum_{s=k}^{d} \binom{d}{s} \binom{d+\varepsilon}{s-1} \binom{s}{k} \\ = \frac{2^\varepsilon}{d} (-1)^k \binom{d}{k} \binom{2d+\varepsilon-k}{d+\varepsilon-k+1}, \end{aligned}$$

where the term with lower index -1 is zero. The coefficient of u^{d-k} in F_d is

$$(-1)^k C_{d+\varepsilon-k} \binom{2d+\varepsilon-k}{k}.$$

After cancelling the common sign, the two expressions have the same factorial form:

$$\frac{2^\varepsilon}{d} \binom{d}{k} \binom{2d+\varepsilon-k}{d+\varepsilon-k+1} = \frac{2^\varepsilon (2d+\varepsilon-k)!}{k!(d-k)!(d+\varepsilon-k+1)!} = C_{d+\varepsilon-k} \binom{2d+\varepsilon-k}{k}.$$

Let $\eta_1 < \cdots < \eta_d$ be the zeros of $\mathcal{E}_{d+\beta+3/2} F_0$. Applying Lemma **??** to F_0 gives $\eta_{i+1} < s_{i+1}$ for $1 \leq i < d$. Let $\zeta_{j,i}$ denote the ith zero of R_j. The relation $F_0 \prec F_1$ and Proposition **??** give

$$s_i < \zeta_{1,i} \leq \zeta_{2,i} \leq \cdots \leq \zeta_{d,i} = r_i \qquad (1 \leq i < d).$$

In particular, $s_i < r_i$ for $1 \leq i < d$. Since $\beta > -1$ and the parameter increments 1 and $\frac{3}{2}$ lie in $[0,2]$, [**?**, Theorem 2.3] applies to $P_d^{(0,\beta)}$ and $P_{d-1}^{(1,\beta+3/2)}$. Their zeros therefore strictly interlace. Since the affine change of variable $z = 2u-1$ is increasing, (**??**) gives $\eta_i < r_i < \eta_{i+1}$ and hence $s_i < r_i < \eta_{i+1} < s_{i+1}$ for $1 \leq i < d$. The remaining inequality $s_d < 1$ follows from $F_0 \prec F_1$, because 1 is the largest zero of F_1. $\square$

Proof of Theorem **??**. We now combine the adjacent comparisons with the endpoint comparison. The cases $n = 0, 1$ are immediate, so assume $n \geq 2$. Equation (**??**) and Lemmas **??** and **??** give $F_0 \prec F_1$. Proposition **??** gives the remaining adjacent interlacings, while Lemma **??** gives $F_0 \prec F_d$. List the zeros of each F_j in nondecreasing order as $\xi_{j,1}, \ldots, \xi_{j,d}$. The adjacent interlacings imply $\xi_{i,k} \leq \xi_{j,k}$ whenever $i < j$, while the endpoint comparison gives $\xi_{d,k} < \xi_{0,k+1}$ for $k < d$. Therefore, for $0 \leq i < j \leq d$ and $1 \leq k < d$,

$$\xi_{i,k} \leq \xi_{j,k} \leq \xi_{d,k} < \xi_{0,k+1} \leq \xi_{i,k+1}, \qquad \xi_{0,d} < 1 = \xi_{1,d} = \cdots = \xi_{d,d}.$$

Hence $F_0, \ldots, F_d$ is interlacing. Moreover,

$$F_j(0) = (-1)^d C_{n-j-d} \binom{n-d}{d} \neq 0 \qquad (0 \leq j \leq d),$$

since $n - j - d = d + \varepsilon - j \geq 0$. Thus $u = 0$ is not a zero of any F_j. The term with $k = d$ in (**??**) also gives

$$[x^d] g_{n,j}(x) = C_{n-j-d} \binom{n-d}{d} > 0,$$

so every $g_{n,j}$ has positive leading coefficient. Finally, the substitution $x = 1 - u^{-1}$ is strictly increasing on $(0, \infty)$. It sends the zero 1 to 0 and every zero in $(0, 1)$ to a negative zero. Hence the same interlacing relations hold for $g_{n,0}, \ldots, g_{n,d}$. $\square$

The above proof implies

$$g_{n,0} \prec g_{n,j} \qquad (1 \leq j \leq \lfloor n/2 \rfloor).$$

The polynomial $g_{n,0}$ has simple negative zeros. For $j \geq 1$, the zero 0 of $g_{n,j}$ is simple, and all its other zeros are simple and negative.

We finish the section with the consequence for nonnegative row combinations; see also Alexandersson [**?**, Theorem 8.1]. The cases $n = 0, 1$ are immediate. Fix an integer $n \geq 2$, put $d = \lfloor n/2 \rfloor$, and let

$$H(x) = \sum_{j=0}^{d} c_j g_{n,j}(x), \qquad c_j \geq 0.$$

Assume that $H \neq 0$. If $c_j = 0$ for every $j \geq 1$, then H is a positive multiple of $g_{n,0}(x)$ and is real-rooted. Otherwise, let

$$\alpha_1 < \cdots < \alpha_d$$

be the zeros of $g_{n,0}$. Since $g_{n,0} \prec g_{n,j}$ for every $j \geq 1$, the values $g_{n,j}(\alpha_i)$ are nonzero and have the same sign, namely

$$\operatorname{sgn}\bigl(g_{n,j}(\alpha_i)\bigr) = (-1)^{d-i+1}.$$

It follows that

$$\operatorname{sgn}\bigl(H(\alpha_i)\bigr) = (-1)^{d-i+1}.$$

Hence H has a zero in every interval (α_i, α_{i+1}) for $1 \leq i < d$. Moreover, $H(\alpha_d) < 0$, while $H(x) > 0$ for all sufficiently large positive x. Since all $g_{n,j}(x)$ have degree d and positive leading coefficient, H also has degree d. The d distinct real zeros found above therefore exhaust its degree, so H is real-rooted.

Proof of Theorem **??**. Equation (**??**) expresses $g(P, x)$ as a nonzero nonnegative linear combination of the nth row because $\gamma_0(P) = 1$. Hence $g(P, x)$ is real-rooted. $\square$

4 Interlacing along fixed-$n-2j$ diagonals

By Section 3, every admissible $g_{n,j}(x)$ has positive leading coefficient and only simple nonpositive real zeros. We also use Xiao's column interlacing

$$g_{n,j}(x) \preceq g_{n+1,j}(x) \qquad (n \geq 2j), \tag{28}$$

which is part of [?, Theorem 1.3].

To compare consecutive terms on the fixed-$n-2j$ diagonals, we first need strictness at the negative zeros. Although (??) gives only weak interlacing, the following recurrence supplies the required strictness:

$$\begin{aligned}(n-j+2)g_{n+1,j}(x) &= (nx+3n-4j+2)g_{n,j}(x) + 2x(1-x)g'_{n,j}(x)\\ &\quad + 2(n-2j)(x-1)g_{n-1,j}(x),\end{aligned} \tag{29}$$

where the last term is omitted when $n = 2j$. For $n \geq 2j+1$, this is [?, Corollary 2.4]. In the boundary case $n = 2j$, the identity follows directly from (??) by comparing coefficients of $(x-1)^k$.

Let $f(x) = g_{n,j}(x)$ and $h(x) = g_{n+1,j}(x)$, and let $r < 0$ be a zero of f. The zero r is simple. If $n = 2j$, then (??) gives

$$(j+2)h(r) = 2r(1-r)f'(r) \neq 0.$$

Assume $n > 2j$ and put $p(x) = g_{n-1,j}(x)$. By (??), $p \preceq f$. Put $D = \deg f$, let $r_1 < \cdots < r_D$ be the zeros of f, and suppose that $r = r_i$. If $p(r_i) \neq 0$, then the interlacing $p \preceq f$ shows that exactly $D-i$ zeros of p lie to the right of r_i. Since p has positive leading coefficient,

$$\operatorname{sgn}\bigl(p(r_i)\bigr) = (-1)^{D-i} = \operatorname{sgn}\bigl(f'(r_i)\bigr).$$

If $p(r_i) = 0$, only the derivative term remains, and it is nonzero. If $p(r_i) \neq 0$, then

$$2r(1-r) < 0, \qquad 2(n-2j)(r-1) < 0,$$

and the two remaining terms on the right side of (??) have the same sign. Hence $h(r) \neq 0$ in either case. Thus the column interlacing is strict at every negative zero of $g_{n,j}$.

The second ingredient is an explicit slope at the origin. Put $y = z + (x-1)z^2$ in (??). At $x = 0$ we have $y = z(1-z)$ and

$$C(z(1-z)) = \frac{1}{1-z}, \qquad C'(z(1-z)) = \frac{1}{(1-2z)(1-z)^2}.$$

Therefore

$$\frac{\partial}{\partial x}\bigl(y^j C(y)\bigr)\Big|_{x=0} = \frac{z^{j+1}(1-z)^{j-2}\bigl(j-(2j-1)z\bigr)}{1-2z}.$$

For $j = 1$ this is $z^2/(1-2z)$. If $j \geq 2$, put $A_j(z) = (1-z)^{j-2}(j-(2j-1)z)$. Since $\deg A_j = j-1$ and $n-j-1 \geq j-1$,

$$[z^{n-j-1}]\frac{A_j(z)}{1-2z} = 2^{n-j-1}A_j(1/2) = 2^{n-2j}.$$

Consequently,

$$g'_{n,j}(0) = 2^{n-2j} \qquad (n \geq 2j,\ j \geq 1). \tag{30}$$

Proof of Theorem **??**. Fix an integer $\ell \geq 0$. It is enough to compare consecutive terms. Write $n = \ell + 2j$ and put

$$f(x) = g_{n,j}(x), \qquad h(x) = g_{n+1,j}(x), \qquad Q(x) = g_{n+2,j+1}(x) = h(x) + (x-1)f(x),$$

where the last identity is (**??**).

First suppose $j = 0$, and put $t = \deg f$. When $t = 0$, the polynomial Q is linear with positive leading coefficient and $Q(0) = 0$, so $f \preceq Q$. Assume $t \geq 1$, and let $\alpha_1 < \cdots < \alpha_t < 0$ be the zeros of f. The strictness at negative zeros established above and the positive leading coefficients give

$$\operatorname{sgn}\bigl(Q(\alpha_i)\bigr) = \operatorname{sgn}\bigl(h(\alpha_i)\bigr) = (-1)^{t-i+1}.$$

Since Q has degree $t+1$ and positive leading coefficient, $Q(x)$ has sign $(-1)^{t+1}$ for all sufficiently negative x, which is opposite to the sign at α_1. Hence Q has one zero to the left of α_1 and one in each interval (α_i, α_{i+1}). Its remaining zero is $Q(0) = 0$. Since $\deg Q = t+1$, these are all its zeros, and hence $f \preceq Q$.

Now suppose $j \geq 1$. Write

$$f(x) = x\widehat{f}(x), \qquad h(x) = x\widehat{h}(x), \qquad Q(x) = x\widehat{Q}(x).$$

Put $t = \deg \widehat{f}$, so $\deg \widehat{Q} = t+1$. All three quotients have positive leading coefficients, and (**??**) gives

$$\widehat{Q}(0) = Q'(0) = 2^{n-2j} > 0.$$

If $t = 0$, then $\widehat{Q}$ is linear and has a negative zero, so $f \preceq Q$. Assume $t \geq 1$, and let $\alpha_1 < \cdots < \alpha_t < 0$ be the zeros of $\widehat{f}$. Since 0 is the common largest zero in (**??**) and the interlacing is strict at every negative zero, removing 0 leaves $\widehat{f}$ and $\widehat{h}$ strictly interlacing. Therefore

$$\operatorname{sgn}\bigl(\widehat{Q}(\alpha_i)\bigr) = \operatorname{sgn}\bigl(\widehat{h}(\alpha_i)\bigr) = (-1)^{t-i+1}.$$

Since $\widehat{Q}$ has degree $t+1$ and positive leading coefficient, $\widehat{Q}(x)$ has sign $(-1)^{t+1}$ for all sufficiently negative x, opposite to its sign at α_1. Thus $\widehat{Q}$ has one zero to the left of α_1 and one in each interval (α_i, α_{i+1}). Since $\widehat{Q}(\alpha_t) < 0 < \widehat{Q}(0)$, it has one further zero in $(\alpha_t, 0)$. Since $\deg \widehat{Q} = t+1$, these are all its zeros. Restoring the common zero 0 gives

$$g_{n,j}(x) \preceq g_{n+2,j+1}(x).$$

This proves the theorem. □

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Declaration of AI usage

During the development of this work, we used ChatGPT 5.6 to verify our proposed strategies and to assist with the computational experiments. Several subtle details of the construction and computations were jointly figured out by ChatGPT and the authors.

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