The ultra log-concavity of Z-polynomials and γ-polynomials of uniform matroids

Siyi Wu[1], Matthew H. Y. Xie[2], Philip B. Zhang[3*]

[1,3]College of Mathematical Science
Tianjin Normal University, Tianjin 300387, P. R. China

[2]College of Science
Tianjin University of Technology, Tianjin 300384, P. R. China

Abstract. Proudfoot, Xu, and Young introduced the Z-polynomial for any matroid and conjectured the polynomial has only real roots. Recently, Ferroni, Nasr, and Vecchi introduced the γ-polynomial of a matroid. In this paper, we prove that both the Z-polynomials and γ-polynomials of uniform matroids are ultra log-concave, which due to Newton's inequality partially supports the real-rootedness conjecture. We also give an alternative formula for the γ-polynomials of uniform matroids. As an application, we use this formula to provide a new proof of the γ-positivity of sparse paving matroids.

Keywords: Log-concavity; Z-polynomial; γ-polynomial; uniform matroid; sparse paving matroid; recurrence relation

AMS Classification 2020: 05A15, 05B35, 33F10

1 Introduction

Given a matroid M, Elias, Proudfoot, and Wakefield [3] introduced the Kazhdan-Lusztig polynomial $P_M(t)$ and Proudfoot, Xu, and Young [13] introduced the Z-polynomial $Z_M(t)$. Quite recently, Ferroni, Nasr, and Vecchi [5] introduced the γ-polynomial $\gamma_M(t)$ of M. These polynomials have been shown to have a deep connection with algebraic geometry, see [1, 3, 11–13]. They are also conjectured to possess further nice properties, one of which we are particularly interested in here is the log-concavity and the real-rootedness [7, 13].

The famous Newton's inequality says that if a polynomial $\sum_{i=0}^{n} a_i t^i$ with real coefficients has only real roots then its coefficients are *ultra log-concave*, i.e.

$$\frac{a_i^2}{\binom{n}{i}^2} \geq \frac{a_{i+1}}{\binom{n}{i+1}} \frac{a_{i-1}}{\binom{n}{i-1}}$$

*Corresponding author. Email: `zhang@tjnu.edu.cn`

Xie is supported by the National Natural Science Foundation of China under Grant No. 12271403. Zhang is supported by the National Natural Science Foundation of China under Grant No. 12171362.

for all $1 \leq i \leq n-1$. We say that a finite sequence $\{a_i\}_{i=0}^n$ is *log-concave* if the consecutive numbers satisfy $a_i^2 \geq a_{i-1}a_{i+1}$ for any $1 \leq i \leq n-1$. It is not hard to see that ultra log-concavity implies log-concavity.

The objective of this paper is to prove the ultra log-concavity of Z-polynomials and γ-polynomials of uniform matroids.

Assume that m and d are positive integers. We denote by $U_{m,d}$ the uniform matroid with rank d on $m+d$ elements. Suppose that

$$Z_{U_{m,d}}(t) = \sum_{i=0}^{d} z_{m,d,i} t^i.$$

Recently, Xie and Zhang [14] confirmed the log-concavity of the Kazhdan-Lusztig polynomials of uniform matroids. In this paper, we shall prove the ultra log-concavity of Z-polynomials of uniform matroids $Z_{U_{m,d}}(t)$.

Theorem 1. *For any positive integers m and d, the sequence $\{z_{m,d,i}\}_{i=0}^d$ is ultra log-concave.*

In this paper, we also study the γ-positivity of uniform matroids. A polynomial $f(t) = \sum_{i=0}^d a_i t^i$ is said to be *palindromic* if $a_i = a_{d-i}$ for any $0 \leq i \leq d$. It is known that if $f(t) \in \mathbb{Z}[t]$ is a palindromic polynomial of degree d, then there exists integer numbers $\gamma_0, \ldots, \gamma_{\lfloor \frac{d}{2} \rfloor}$ such that

$$f(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i (1+t)^{d-2i}. \tag{1}$$

Ferroni, Nasr, and Vecchi [5] defined the *γ-polynomial* associated to $f(t)$ by

$$\gamma_f(t) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \gamma_i t^i.$$

A palindromic polynomial $f(t)$ of degree d is defined to be *γ-positive* if all the coefficients of $\gamma_f(t)$ are non-negative. Since the Z-polynomial of a matroid is palindromic, Ferroni, Nasr, and Vecchi [5] introduced the *γ-polynomial* of a matroid M to be the polynomial

$$\gamma_M(t) = \gamma(Z_M, t).$$

They conjectured that for any matroid M the polynomial $\gamma_M(t)$ has non-negative coefficients and confirmed it for uniform matroids and sparse paving matroids. This conjecture has been completely resolved by Ferroni, Matherne, Stevens, and Vecchi [4] recently.

Lemma 2 ([5, Theorem 5.9])**.** *For any positive integers m and d, the uniform matroid $U_{m,d}$ is γ-positive. In addition, the coefficient of each degree $i > 0$ in $\gamma_{U_{m,d}}(t)$ is given as follows:*

$$r_{m,d,i} = \frac{1}{d-i}\binom{d-i}{i}\sum_{j=i}^{d-1}(d-j)\binom{j-1}{i-1}\binom{m+j-1}{j}.$$

In this paper, we give an alternative expression for $r_{m,d,i}$ and a mysterious connection with the Kazhdan-Lusztig polynomials of $U_{m,d}$.

Theorem 3. *For any positive integers m, d and $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we have*

$$r_{m,d,i} = \frac{d!}{i!(d-2i)!(i-1)!}\sum_{h=0}^{m-1}\frac{\binom{h+d}{h}}{(h+i)(h+i+1)}. \tag{2}$$

Based on (2), we prove the following result.

Theorem 4. *For any positive integers m and d, the polynomial $\gamma_{U_{m,d}}(t)$ is ultra log-concave.*

The outline of this paper is organized as follows. We prove Theorem 1 in Section 2 by using a computer algebra system. In Section 3, we prove Theorem 3 and Theorem 4. In Section 4, we give a new proof of the γ-positivity for sparse paving matroids as an application of Theorem 3.

2 Proof of Theorem 1

We prove Theorem 1 by using a computer algebra system through the following ideas. Recall that Gao, Lu, Xie, Yang, and Zhang [6] gave a formula for $z_{m,d,i}$ as follows,

$$z_{m,d,i} = \frac{\binom{d+m}{i+m}\binom{d+m}{i}}{\binom{d+m}{m}}\sum_{h=0}^{m-1}\frac{i(h-m+1)+m}{(h+1)m}\binom{h+i-1}{h}\binom{d+h-i}{h}. \tag{3}$$

Let

$$p_{d,i} = \frac{\binom{d+m}{i+m}\binom{d+m}{i}}{\binom{d+m}{m}}$$

and

$$q_{n,i} = \sum_{h=0}^{m-1}\frac{i(h-m+1)+m}{(h+1)m}\binom{h+i-1}{h}\binom{n+h}{h}.$$

For the convenience, we ignore the m index in $p_{d,i}$ and $q_{n,i}$. Consequently,

$$z_{m,d,i} = p_{d,i}q_{d-i,i}.$$

As the ultra log-concavity of $p_{d,i}$ is easy to check, the key ingredient of our proof is the following lemma, which confirms the log-concavity of $q_{d-i,i}$.

Lemma 5. *For any positive integers m, i, d with $2 \le i \le d-2$, we have*

$$q_{d-i,i}^2 \ge q_{d-i-1,i+1} q_{d-i+1,i-1}.$$

Letting $n = d-i$, we may rewrite Lemma 5 as follows: For any positive integers $i \ge 2$ and $n \ge 2$, we have

$$q_{n,i}^2 \ge q_{n-1,i+1} q_{n+1,i-1}. \tag{4}$$

Our proof of (4) is similar to the proof of log-concavity of Kazhdan-Lusztig polynomials of uniform matroids by Xie and Zhang [14], which is inspired by Kauers and Paule's computer proof of Moll's log-concavity conjecture [9].

The rest of this section is organized as follows. We first give some recurrence relations of $q_{n,i}$ in Subsection 2.1. We next estimate upper and lower bounds of $\frac{q_{n,i}}{q_{n-1,i}}$ in Subsection 2.2. In Subsection 2.3, we convert (4) into three inequalities and prove them respectively. Finally, we complete the proof of Lemma 5 and Theorem 1 .

2.1 Recurrence relations of $q_{n,i}$

In this subsection we give some recurrence relations of $q_{n,i}$, which will be used later.

In order to obtain the required recurrence relations we use the package **HolonomicFunctions**[†] by Koutschan [10] for **Mathematica**.

Lemma 6. *For any positive integers m, i and n, we have*

$$q_{n+1,i} = \frac{-(n-1)(m+n)q_{n-1,i} + \left(m(n-1) + 2n^2 + i(m+n-1)\right) q_{n,i}}{n(n+i+1)}, \tag{5}$$

$$q_{n-1,i+1} = \frac{(1+i-n)q_{n-1,i} + (n-1)q_{n,i}}{i}, \tag{6}$$

$$q_{n-1,i-1} = \frac{\left(i^2 + m(n-2) + (n-1)^2 + i(m+n-2)\right) q_{n-1,i} - (n-1)(i+n)q_{n,i}}{(i-1)(m+i)}, \tag{7}$$

$$q_{n,i+1} = \frac{-(n-1)(m+n)q_{n-1,i} + \left(i^2 + i(m+n) + (n-1)(m+n)\right) q_{n,i}}{i(n+i+1)}. \tag{8}$$

[†] The HolonomicFunctions package can be downloaded at https://www3.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html.

Proof. The command **Annihilator[*expr*]** can compute annihilating operators for the expression *expr*.

In[1]:= $\mathbf{ann = Annihilator[Sum[\frac{\boldsymbol{i(h-m+1)+m}}{\boldsymbol{(h+1)m}}Binomial[\boldsymbol{i-1+h,h}]Binomial[\boldsymbol{n+h,h}],\{\boldsymbol{h,0,m-1}\}],\{\boldsymbol{S[i],S[n]}\}]}$

Out[1]= $\{-iS_i+nS_n+(i-n),(1+n)(2+i+n)S_n^2+(-2-i(m+n)-n(4+m+2n))S_n+n(1+m+n)\}$

Here S_n (respectively S_i) denotes the forward shift in n (respectively i).

We use the **OreReduce** command of Koutschan's package to prove the required recurrence relations. We take (6) as an example to demonstrate how **OreReduce** works. We first replace $n-1$ with n in (6). Thus, it follows that

$$iq_{n,i+1} - nq_{n+1,i} - (i-n)q_{n,i} = 0, \tag{9}$$

We need to prove that the Ore polynomial $iS[i] - nS[n] - (i-n)$ modulo the ideal **ann** is zero. This can be verified by the following lines.

In[2]:= $\mathbf{OreReduce}\,[\boldsymbol{iS[i]-nS[n]-(i-n),ann}]$

Out[2]= 0

Similarly, we can prove the equations (5), (7) and (8) by the following lines.

In[3]:= $\mathbf{OreReduce}\,[\boldsymbol{(n+1)(n+i+2)S[n]^2+n(m+n+1)-(mn+2(n+1)^2+i(m+n))\,S[n],ann}]$

Out[3]= 0

In[4]:= $\mathbf{OreReduce}\,[\boldsymbol{i(m+i+1)-((i+1)^2+m(n-1)+n^2+(i+1)(m+n-1))S[i]+n(i+n+2)S[n]S[i],ann}\,]$

Out[4]= 0

In[5]:= $\mathbf{OreReduce}\,[\boldsymbol{i(n+i+2)S[n]S[i]+n(m+n+1)-(i^2+i(m+n+1)+n(m+n+1))S[n],ann}\,]$

Out[5]= 0

Now all the required recurrence relations have been obtained. This completes the proof. □

2.2 Bounds of $\frac{q_{n,i}}{q_{n-1,i}}$

In this subsection we estimate the upper and lower bounds of $\frac{q_{n,i}}{q_{n-1,i}}$.

To present the bounds of $\frac{q_{n,i}}{q_{n-1,i}}$, we let

$$X(n,i) = \frac{1 - 2m - 3n + mn + 2n^2 + i(m+n-3) + \sqrt{\Delta_1(n,i)}}{2(n-1)(i+n)},$$

and

$$Y(n,i) = \frac{i(m+n-2) + (n-1)(m+2n+1) + \sqrt{\Delta_2(n,i)}}{2(n-1)(n+i+1)},$$

where

$$\Delta_1(n,i) = 2m(n-1)\left(i^2 + (i-1)m + 1\right) + (n-1)^2(i+m-1)^2 + (m - i(m-2))^2$$

and

$$\Delta_2(n,i) = i^2(m+n-2)^2 + 2i(m-1)(n-1)(m+n+2) + (m-1)^2(n-1)^2.$$

Lemma 7. *For any positive integers m, i, and $n \geq 2$, we have*

$$X(n,i) \leq \frac{q_{n,i}}{q_{n-1,i}} \leq Y(n,i). \tag{10}$$

Proof. Fixing $i \geq 1$, we prove this lemma by induction on n. For the case $n = 2$, we need to show

$$X(2,i) \leq \frac{q_{2,i}}{q_{1,i}} \leq Y(2,i).$$

By the definition of $q_{n,i}$, we obtain

$$q_{1,i} = \frac{i+m}{i(i+1)}\binom{i+m-1}{m}$$

and

$$q_{2,i} = \frac{(i+m)(im+2)}{i(i+1)(i+2)}\binom{i+m-1}{m}.$$

Thus

$$\frac{q_{2,i}}{q_{1,i}} = \frac{im+2}{i+2}.$$

On the other hand, $X(2,i)$ and $Y(2,i)$ can be expressed as follows,

$$X(2,i) = \frac{i(m-1) + 3 + \sqrt{i^2\left(m^2 - 2m + 5\right) + 2i(3m-1) + 1}}{2(i+2)}$$

and

$$Y(2,i) = \frac{im + m + 5 + \sqrt{(im+m+5)^2 - 4(i+3)(m+2)}}{2(i+3)}.$$

Thus, we just need to prove

$$\frac{i(m-1)+3+\sqrt{i^2\left(m^2-2m+5\right)+2i(3m-1)+1}}{2(i+2)} \le \frac{im+2}{i+2} \tag{11}$$

$$\le \frac{im+m+5+\sqrt{(im+m+5)^2-4(i+3)(m+2)}}{2(i+3)}. \tag{12}$$

This inequality can be easily proved directly, but we use a computer algebra to prove it here. Here, we introduce the Mathematica command **CylindricalDecomposition** [8]. The Cylindrical Algebraic Decomposition (CAD) algorithm was invented by Collins [2] in order to do quantifier elimination over the reals: given a quantified formula, it finds a formula without quantifiers which is equivalent over the reals to the input formula.

In[6]:= $\boldsymbol{\Delta_1[n\_,i\_] := 2m(n-1)\left(i^2+(i-1)m+1\right)+(n-1)^2(i+m-1)^2+(m-i(m-2))^2;}$

In[7]:= $\boldsymbol{X[n\_,i\_] := \frac{1-2m-3n+mn+2n^2+i(m+n-3)+\sqrt{\Delta_1[n,i]}}{2(n-1)(i+n)};}$

In[8]:= $\boldsymbol{\Delta_2[n\_,i\_] := i^2(m+n-2)^2+2i(m-1)(n-1)(m+n+2)+(m-1)^2(n-1)^2;}$

In[9]:= $\boldsymbol{Y[n\_,i\_] := \frac{i(m+n-2)+(n-1)(m+2n+1)+\sqrt{\Delta_2[n,i]}}{2(n-1)(n+i+1)};}$

In[10]:= **CylindricalDecomposition [Implies [** $\boldsymbol{m \ge 1 \&\& i \ge 1, X[2,i] \le (im+2)/(i+2) \le Y[2,i]}$ **] ,** $\boldsymbol{\{m,i\}}$ **]**

Out[10]= True

Therefore, the desired inequality holds for $n=2$.

Assume that the inequality holds for the general n, namely,

$$X(n,i) \le \frac{q_{n,i}}{q_{n-1,i}} \le Y(n,i).$$

We proceed to prove the desired inequality holds for $n+1$ as well. It follows from (5) that

$$\frac{q_{n+1,i}}{q_{n,i}} = \frac{-(n-1)(m+n)q_{n-1,i}}{n(n+i+1)q_{n,i}} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)}. \tag{13}$$

Together with $-(n-1)<0$ we obtain

$$\frac{-(n-1)(m+n)}{n(n+i+1)}\frac{1}{X(n,i)} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)} \le \frac{q_{n+1,i}}{q_{n,i}} \tag{14}$$

$$\le \frac{-(n-1)(m+n)}{n(n+i+1)}\frac{1}{Y(n,i)} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)}. \tag{15}$$

So we need to show that the left side of (14) is more than or equal to $X(n+1,i)$ and the right side of (15) is less than or equal to $Y(n+1,i)$.

In[11]:= $\frac{-(n-1)(m+n)}{n(n+i+1)X[n,i]} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)} \geq X[n+1,i];$

In[12]:= CylindricalDecomposition [Implies [$m \geq 1$&&$n \geq 2$&&$i \geq 1$, %] , {i, n, m}]

Out[12]= True

In[13]:= $\frac{-(n-1)(m+n)}{n(n+i+1)Y[n,i]} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)} \leq Y[n+1,i];$

In[14]:= CylindricalDecomposition [Implies [$m \geq 1$&&$n \geq 2$&&$i \geq 1$, %] , {i, n, m}]

Out[14]= True

This completes the proof. □

2.3 Proof of Lemma 5 and Theorem 1

To prove Lemma 5, we divide the inequality (4) into

$$q_{n,i}^2 \geq q_{n,i+1}q_{n,i-1} \tag{16}$$

and

$$\frac{q_{n,i+1}}{q_{n-1,i+1}} \geq \frac{q_{n,i}}{q_{n-1,i}} \geq \frac{q_{n+1,i}}{q_{n,i}} \geq \frac{q_{n+1,i-1}}{q_{n,i-1}}. \tag{17}$$

First, we prove (16).

Lemma 8. *For any positive integers $i \geq 2, n \geq 2$ and m, we have $q_{n,i}^2 \geq q_{n,i+1}q_{n,i-1}$.*

Proof. For the convenience of notation, we prove $q_{n-1,i}^2 \geq q_{n-1,i+1}q_{n-1,i-1}$ for $n \geq 2$.

Through the recurrence relations (6) and (7), we get that

$$q_{n-1,i}^2 - q_{n-1,i+1}q_{n-1,i-1} = \frac{(n-1)q_{n-1,i}^2}{i(i-1)(i+m)} f_{n,i}\left(\frac{q_{n,i}}{q_{n-1,i}}\right).$$

Let

$$\begin{aligned} f_{n,i}(x) = &(n-1)(i+n)x^2 - (i(m+n-3)+(m-3)n-2m+2n^2+1)x \\ &+ (-i+m(-2+n)+(-1+n)^2). \end{aligned}$$

By a direct computation, we find that $f_{n,i}(x)$ is just $\Delta_1(n,i)$ defined in Section 2.2. Recall that

$$\Delta_1(n,i) = 2m(n-1)\left(i^2+(i-1)m+1\right) + (n-1)^2(i+m-1)^2 + (m-i(m-2))^2.$$

Since $n \geq 2$ and $i \geq 2$, it is obvious to see that $\Delta_1(n,i) > 0$. Therefore, $f_{n,i}(x)$ has two distinct zeros, which are

$$x_1(n,i) = \frac{1 - 2m - 3n + mn + 2n^2 + i(m+n-3) - \sqrt{\Delta_1(n,i)}}{2(n-1)(i+n)},$$

$$x_2(n,i) = \frac{1 - 2m - 3n + mn + 2n^2 + i(m+n-3) + \sqrt{\Delta_1(n,i)}}{2(n-1)(i+n)}.$$

We find that $x_2(n,i) = X(n,i)$. Then since the leading coefficient of $f_{n,i}(x)$ is positive, it follows from Lemma 7 that

$$x_2(n,i) \leq \frac{q_{n,i}}{q_{n-1,i}},$$

which completes the proof. □

Lemma 9. *For any positive integers m and $n \geq 2$, we have*

$$\frac{q_{n,i+1}}{q_{n-1,i+1}} \geq \frac{q_{n,i}}{q_{n-1,i}} \quad (i \geq 1), \tag{18}$$

$$\frac{q_{n+1,i}}{q_{n,i}} \geq \frac{q_{n+1,i-1}}{q_{n,i-1}} \quad (i \geq 2). \tag{19}$$

Proof. First, we prove the inequality (18), namely

$$q_{n,i+1}q_{n-1,i} - q_{n,i}q_{n-1,i+1} \geq 0.$$

The inequality(19) can be deduced from (18).

By the recurrence relations (6) and (8), we have

$$\begin{aligned} & i(n+i+1)(q_{n,i+1}q_{n-1,i} - q_{n,i}q_{n-1,i+1}) \\ = & -(n-1)(n+i+1)q_{n,i}^2 + (i(m+n-2) + (n-1)(m+2n+1))q_{n,i}q_{n-1,i} \\ & -(n-1)(m+n)q_{n-1,i}^2. \end{aligned}$$

Let

$$g_{n,i}(x) = -(n-1)(n+i+1)x^2 + (i(m+n-2) + (n-1)(m+2n+1))x - ((n-1)(m+n)).$$

Thus

$$i(n+i+1)(q_{n,i+1}q_{n-1,i} - q_{n,i}q_{n-1,i+1}) = q_{n-1,i}^2 g_{n,i}\left(\frac{q_{n,i}}{q_{n-1,i}}\right).$$

Since $i(n+i+1) > 0$ and $q_{n-1,i}^2 > 0$, we just need to show that $g_{n,i}(\frac{q_{n,i}}{q_{n-1,i}}) \geq 0$. We can directly verify that the discriminant of $g_{n,i}(x)$ is equal to $\Delta_2(n,i)$ defined in Section 2.2. Recall that

$$\Delta_2(n,i) = i^2(m+n-2)^2 + 2i(m-1)(n-1)(m+n+2) + (m-1)^2(n-1)^2,$$

which is manifestly positive since $m \geq 1$ and $n \geq 2$. Then $g_{n,i}(x)$ has two distinct zeros $y_1(n,i)$ and $y_2(n,i)$, which are

$$y_1(n,i) = \frac{i(m+n-2)+(n-1)(m+2n+1)-\sqrt{\Delta_2(n,i)}}{2(n-1)(n+i+1)},$$

$$y_2(n,i) = \frac{i(m+n-2)+(n-1)(m+2n+1)+\sqrt{\Delta_2(n,i)}}{2(n-1)(n+i+1)}.$$

Since the leading coefficient of $g_{n,i}(x)$ is negative, to make $g_{n,i}(\frac{q_{n,i}}{q_{n-1,i}}) \geq 0$ hold, we next prove

$$y_1(n,i) \leq \frac{q_{n,i}}{q_{n-1,i}} \leq y_2(n,i).$$

Here $y_2(n,i) = Y(n,i)$, and we have already proved that $X(n,i) \leq \frac{q_{n,i}}{q_{n-1,i}} \leq Y(n,i)$ in Lemma 7. Hence, we get $\frac{q_{n,i}}{q_{n-1,i}} \leq y_2(n,i)$. It remains to show that $y_1(n,i) \leq X(n,i)$.

In[15]:= $\boldsymbol{y_1[n\_, i\_] := \frac{i(m+n-2)+(n-1)(m+2n+1)-\sqrt{\Delta_2[n,i]}}{2(n-1)(n+i+1)};}$

In[16]:= $\boldsymbol{y_1[n,i] \leq X[n,i];}$

In[17]:= **CylindricalDecomposition [Implies[$m \geq 1$&&$n \geq 2$&&$i \geq 1$, %], {i, n, m}]**

Out[17]= True

This completes the proof. □

Lemma 10. *For any positive integers m, i, $n \geq 2$, we have*

$$\frac{q_{n,i}}{q_{n-1,i}} \geq \frac{q_{n+1,i}}{q_{n,i}}. \tag{20}$$

Proof. By (13), the desired inequality (20) is converted to

$$\frac{q_{n,i}}{q_{n-1,i}} \geq \frac{-(n-1)(m+n)q_{n-1,i}}{n(n+i+1)q_{n,i}} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)}.$$

Multiplying both sides of the above inequality by $\frac{q_{n,i}}{q_{n-1,i}}$, we need to prove that

$$\left(\frac{q_{n,i}}{q_{n-1,i}}\right)^2 \geq \frac{-(n-1)(m+n)}{n(n+i+1)} + \frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)}\left(\frac{q_{n,i}}{q_{n-1,i}}\right).$$

Let

$$z_1(n,i) = \frac{m(n-1)+2n^2+i(m+n-1)-\sqrt{\Delta_3(n,i)}}{2n(n+i+1)}$$

and

$$z_2(n,i)=\frac{m(n-1)+2n^2+i(m+n-1)+\sqrt{\Delta_3(n,i)}}{2n(n+i+1)}$$

be the two real zeros of the equation

$$x^2-\frac{(n-1)(m+n)}{n(n+i+1)}+\frac{m(n-1)+2n^2+i(m+n-1)}{n(n+i+1)}x=0,$$

where $\Delta_3(n,i)=i^2(m+n-1)^2+2im\left(m(n-1)+n^2+1\right)+(-mn+m+2n)^2$.

Recall that, we have proven $\frac{q_{n,i}}{q_{n-1,i}}\geq X(n,i)$. It remains to show that $X(n,i)\geq z_2(n,i)$. This follows from the following lines.

In[18]:= $\boldsymbol{\Delta_3[n\_,i\_]:=i^2(m+n-1)^2+2im\left(m(n-1)+n^2+1\right)+(m+2n-mn)^2;}$

In[19]:= $\boldsymbol{z_2[n\_,i\_]:=\frac{m(n-1)+2n^2+i(m+n-1)+\sqrt{\Delta_3[n,i]}}{2n(n+i+1)};}$

In[20]:= $\boldsymbol{X[n,i]\geq z_2[n,i];}$

In[21]:= **CylindricalDecomposition [Implies[$m\geq 1$&&$n\geq 2$&&$i\geq 1$, %], {i,n,m}]**

Out[21]= True

This completes the proof. □

Now, we prove the main result of this section.

Proof of Lemma 5. We prove this lemma by verifying the inequality (4). By Lemma 9, we have

$$\frac{q_{n,i+1}}{q_{n-1,i+1}}\geq\frac{q_{n,i}}{q_{n-1,i}}$$

and

$$\frac{q_{n+1,i}}{q_{n,i}}\geq\frac{q_{n+1,i-1}}{q_{n,i-1}}.$$

By Lemma 10, we can get

$$\frac{q_{n,i}}{q_{n-1,i}}\geq\frac{q_{n+1,i}}{q_{n,i}}.$$

Therefore, we have

$$\frac{q_{n,i+1}}{q_{n-1,i+1}}\geq\frac{q_{n,i}}{q_{n-1,i}}\geq\frac{q_{n+1,i}}{q_{n,i}}\geq\frac{q_{n+1,i-1}}{q_{n,i-1}}.$$

Then

$$\frac{q_{n,i+1}}{q_{n-1,i+1}}\geq\frac{q_{n+1,i-1}}{q_{n,i-1}},$$

that is

$$q_{n,i+1}q_{n,i-1}\geq q_{n-1,i+1}q_{n+1,i-1}.$$

By Lemma 8, we have

$$q_{n,i}^2 \geq q_{n,i+1}q_{n,i-1}.$$

Hence, together with these two inequalities we get

$$q_{n,i}^2 \geq q_{n-1,i+1}q_{n+1,i-1}.$$

This completes the proof of Lemma 5. □

We proceed to prove Theorem 1.

Proof of Theorem 1. We first prove the ultra log-concavity of $p_{d,i}$, which is equivalent to the following inequality,

$$\frac{(d-i)i\, p_{d,i}^2}{(d-i+1)(i+1)\, p_{d,i+1}p_{d,i-1}} = \frac{(m+i+1)(d-i+m+1)}{(m+i)(d-i+m)} \geq 1.$$

By Lemma 5, we have

$$q_{d-i,i}^2 \geq q_{d-i-1,i+1}q_{d-i+1,i-1}$$

for $2 \leq i \leq d-2$. Since $z_{m,d,i} = p_{d,i}q_{d-i,i}$, we obtain the ultra log-concavity of $z_{m,d,i}$ for $2 \leq i \leq d-2$.

It suffices to prove the ultra log-concavity of $z_{m,d,i}$ for $i = 1$ and $i = d-1$. As the Z-polynomial is a palindromic polynomial, i.e. $z_{m,d,i} = z_{m,d,d-i}$ for any $0 \leq i \leq d$, we remain to show that

$$\left(\frac{z_{m,d,1}}{\binom{d}{1}}\right)^2 \geq \frac{z_{m,d,2}}{\binom{d}{2}}\frac{z_{m,d,0}}{\binom{d}{0}}.$$

By using Gosper's algorithm or Mathematica, it is easy to see that

$$z_{m,d,0} = 1, \quad z_{m,d,1} = \binom{d+m}{m+1}, \quad z_{m,d,2} = \frac{(d-2)m+2}{2}\binom{d+m}{m+2}.$$

Let

$$\phi(d,m) = \frac{z_{m,d,1}^2}{z_{m,d,0}z_{m,d,2}} \cdot \frac{\binom{d}{2}}{\binom{d}{1}^2} = \frac{(m+2)(d+m)!}{d!(m+1)!((d-2)m+2)}.$$

We have $\phi(d,0) = 1$ and

$$\frac{\phi(d,m+1)}{\phi(d,m)} = \frac{(m+3)(d+m+1)((d-2)m+2)}{(m+2)^2((d-2)m+d)} \geq 1$$

by the following lines.

```
In[22]:= CylindricalDecomposition[ForAll[{m, d}, m ≥ 1&&d ≥ 2,
    (m + 3)(d + m + 1)((d − 2)m + 2) / ((m + 2)^2((d − 2)m + d)) ≥ 1], {d, m}]

Out[22]= True
```

By induction on m, we show that $\phi(d,m) \geq 1$ holds for any nonnegative integer m. This completes the proof. □

3 The ultra log-concavity of the γ-polynomials of uniform matroids

In this section, we shall first prove Theorem 3, which gives an alternative formula for $r_{m,d,i}$. Then, we prove the ultra log-concavity of the γ-polynomials of uniform matroids.

Proof of Theorem 3. In order to prove (2), it suffices to prove

$$\frac{1}{d-i}\binom{d-i}{i}\sum_{j=i}^{d-1}(d-j)\binom{j-1}{i-1}\binom{m+j-1}{j} = \frac{d!}{i!(d-2i)!(i-1)!}\sum_{h=0}^{m-1}\frac{\binom{h+d}{h}}{(h+i)(h+i+1)},$$

which is equivalent to

$$\sum_{j=i}^{d-1}\frac{(d-j)(d-i-1)!(j+m-1)!}{j(m-1)!(j-i)!} = \sum_{h=0}^{m-1}\frac{(d+h)!}{h!(h+i)(h+i+1)}. \tag{21}$$

Let $L(m)$ and $R(m)$ be the left side and the right side of (21) respectively. It is obvious to see that $R(1) = \frac{d!}{i(i+1)}$ and $R(m+1) - R(m) = \frac{(d+m)!}{m!(m+i)(m+i+1)}$. We shall prove the sequence $L(m)$ have the same initial value and difference with respect to m.

For the initial case $m = 1$, we can easily check

$$\sum_{j=i}^{d-1}\frac{(d-j)(d-i-1)!(j-1)!}{(j-i)!} = \frac{d!}{i(i+1)}$$

holds by using Gosper's algorithm or Mathematica.

Next we prove $L(m+1) - L(m) = \frac{d+m)!}{m!(m+i)(m+i+1)}$ via the following line.

```
In[23]:= Annihilator[Sum[((d − j)(d − i − 1)!(j + m − 1)!) / ((j(m − 1)!(j − i)!)), {j, i, d − 1}], {S[m]},
    Inhomogeneous → True];
```

In[24]:= **FullSimplify**[%]

Out[24]= $\{\{S[m]-1\},\{-\frac{\text{Gamma}[1+d+m]}{(i+m)(1+i+m)\text{Gamma}[1+m]}\}\}$

This completes the proof of (21). □

Proof of Theorem 4. According to the definition of the ultra log-concavity, it suffices to prove that

$$\left(\frac{r_{m,d,i}}{\binom{d}{i}}\right)^2 \geq \frac{r_{m,d,i+1}}{\binom{d}{i+1}}\frac{r_{m,d,i-1}}{\binom{d}{i-1}}.$$

Recall that Xie and Zhang [14, p2] studied the following numbers

$$a_{d,i} = \frac{1}{(i-1)!}\sum_{h=0}^{m-1}\frac{\binom{h+d}{h}}{(h+i)(h+i+1)},$$

which are used to express the coefficients of Kazhdan-Lusztig polynomials of uniform matroids.

By (2), we have

$$\frac{r_{m,d,i}}{\binom{d}{i}} = \frac{d!}{\binom{d}{i}i!(d-2i)!}a_{d,i}.$$

Thus we shall prove

$$\left(\frac{d!}{i!(d-2i)!\binom{d}{i}}a_{d,i}\right)^2 \geq \frac{d!}{(i+1)!(d-2i-2)!\binom{d}{i+1}}a_{d,i+1}\frac{d!}{(i-1)!(d-2i+2)!\binom{d}{i-1}}a_{d,i-1},$$

which is equivalent to

$$\frac{(d-2i+1)(d-2i+2)(d-i)}{(d-2i-1)(d-2i)(d-i+1)}a_{d,i}^2 \geq a_{d,i+1}a_{d,i-1}.$$

Since

$$\begin{aligned}
&(d-2i+1)(d-2i+2)(d-i)-(d-2i-1)(d-2i)(d-i+1)\\
=&3d^2+d(3-8i)+4(i-1)i\\
\geq&12i^2+2i(3-8i)+4(i-1)i\\
=&2i\geq 0,
\end{aligned}$$

the desired inequality follows from the log-concavity of $(a_{d,i})_i$, namely,

$$a_{d,i}^2 \geq a_{d,i+1}a_{d,i-1},$$

which was proved by Xie and Zhang [14, Lemma 7]. This completes the proof. □

4 The γ-positivity of sparse paving matroids revised

In this section, we use our new formula to give a new proof of the γ-positivity of sparse paving matroids.

A matroid M of rank d is *sparse paving* if and only if each d-subset of $E(M)$ is either a basis or a circuit-hyperplane.

Lemma 11 ([5, Proposition 5.14])**.** *If M is a sparse paving matroid of rank d and cardinality $m+d$ having exactly λ circuit-hyperplanes, then*

$$\gamma_M(t) = \gamma_{U_{m,d}}(t) - \lambda\, g_{d,d}(t), \tag{22}$$

where the number of circuit-hyperplanes λ of M satisfies

$$\lambda \leq \binom{m+d}{d} \min\left\{\frac{1}{d+1}, \frac{1}{m+1}\right\} \tag{23}$$

and

$$g_{d,d}(t) = \sum_{i=1}^{\lfloor \frac{d}{2} \rfloor} \frac{2}{d-i-1} \binom{d-i-1}{i-1} \binom{d-1}{i+1} t^i. \tag{24}$$

Now that we have a new expression for $r_{m,d,i}$, we next use it to reprove the γ-positivity for sparse paving matroids.

Theorem 12 ([5, Theorem 5.15])**.** *Sparse paving matroids are γ-positive.*

Proof. We will assume throughout the proof that M is a sparse paving matroid of rank d and cardinality $m+d$ having exactly λ circuit-hyperplanes. By (22), we have

$$\gamma_M(t) = \gamma_{U_{m,d}}(t) - \lambda \cdot g_{d,d}(t).$$

Let us fix $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$. Proving that $[t^i]\, \gamma_M(t)$ is non-negative amounts to show that

$$r_{m,d,i} = \left[t^i\right] \gamma_{U_{m,d}}(t) \geq \lambda \left[t^i\right] g_{d,d}(t).$$

Let $c_{m,d} = \max(m,d) + 1$. By (23), we obtain that $\lambda \leq \frac{1}{c_{m,d}}\binom{m+d}{d}$. Then it suffices to prove

$$\frac{d!}{i!(d-2i)!(i-1)!} \sum_{h=0}^{m-1} \frac{\binom{d+h}{h}}{(h+i)(h+i+1)} \geq \frac{2}{c_{m,d}(d-i-1)} \binom{m+d}{d} \binom{d-i-1}{i-1} \binom{d-1}{i+1},$$

which can be reduced to

$$\sum_{h=0}^{m-1} \frac{(d+h)!}{(h+i)(h+i+1)h!} \geq \frac{2(m+d)!}{c_{m,d} m!(i+1)d}.$$

This can be obtained from the following inequality

$$\frac{(m+d)!}{(m+i)(m+i+1)m!} \geq \frac{2(m+d+1)!}{c_{m+1,d}(m+1)!(i+1)d} - \frac{2(m+d)!}{c_{m,d}m!(i+1)d},$$

which is equivalent to

$$\frac{d(i+1)}{2(m+i)(m+i+1)} \geq \frac{m+d+1}{c_{m+1,d}(m+1)} - \frac{1}{c_{m,d}}. \tag{25}$$

We prove (25) by using the Cylindrical Algebraic Decomposition algorithm.

In[25]:= $\mathbf{CylindricalDecomposition}\left[\mathbf{ForAll}\left[\{m,d,i\}, m \geq 1 \&\& i \leq d/2 \&\& (i \geq 2 \| i = 1), \frac{d(i+1)}{2(m+i)(m+i+1)} \geq \frac{m+d+1}{\mathrm{Max}[m+1,d](m+1)} - \frac{1}{\mathrm{Max}[m,d]}\right], \{d,i,m\}\right]$

Out[25]= True

This completes the proof. □

Remark 13. *We write $i \geq 1$ as $i \geq 2$ or $i = 1$ in* **In**[22]*, since the inequality* (25) *does not hold for $m = 2, d = 3, i = 3/2$.*

References

[1] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang, Singular Hodge theory for combinatorial geometries, *arXiv:2010.06088*.

[2] G. E. Collins, Quantifier elimination for real closed fields by cylindrical algebraic decomposition, in: *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975)*, Lecture Notes in Comput. Sci., Vol. 33, Springer, Berlin1975, 134–183.

[3] B. Elias, N. Proudfoot, and M. Wakefield, The Kazhdan-Lusztig polynomial of a matroid, *Adv. Math.*, **299** (2016), 36–70.

[4] L. Ferroni, J. P. Matherne, M. Stevens, and L. Vecchi, Hilbert-poincaré series of matroid chow rings and intersection cohomology, *arXiv:2212.03190*.

[5] L. Ferroni, G. D. Nasr, and L. Vecchi, Stressed hyperplanes and Kazhdan-Lusztig gamma-positivity for matroids, *Int. Math. Res. Not.*, (2023), 20883–20942.

[6] A. L. L. Gao, L. Lu, M. H. Y. Xie, A. L. B. Yang, and P. B. Zhang, The Kazhdan-Lusztig polynomials of uniform matroids, *Adv. in Appl. Math.*, **122** (2021), 102117.

[7] K. Gedeon, N. Proudfoot, and B. Young, Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures, *Sém. Lothar. Combin.*, **78B** (2017), Art. 80.

[8] M. Kauers, How to use cylindrical algebraic decomposition, *Sém. Lothar. Combin.*, **65** (2011), Art. B65a.

[9] M. Kauers and P. Paule, A computer proof of Moll's log-concavity conjecture, *Proc. Amer. Math. Soc.*, **135** (2007), 3847–3856.

[10] C. Koutschan, Advanced applications of the holonomic systems approach, Doctoral Thesis, Research Institute for Symbolic Computation(RISC), Johannes Kepler University, Linz, Austria, 2009.

[11] N. Proudfoot, Equivariant Kazhdan-Lusztig polynomials of q-niform matroids, *Algebraic Combin.*, **2** (2019), 613–619.

[12] N. Proudfoot, M. Wakefield, and B. Young, Intersection cohomology of the symmetric reciprocal plane, *J. Algebraic Combin.*, **43** (2016), 129–138.

[13] N. Proudfoot, Y. Xu, and B. Young, The Z-polynomial of a matroid, *Electron. J. Comb.*, **25** (2018), P1.26.

[14] M. H. Y. Xie and P. B. Zhang, The log-concavity of Kazhdan-Lusztig polynomials of uniform matroids, *J. Syst. Sci. Complex*, 36 (2023), 117–128..