Distributions of statistics on separable permutations

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Abstract. We derive functional equations for distributions of six classical statistics (ascents, descents, left-to-right maxima, right-to-left maxima, left-to-right minima, and right-to-left minima) on separable and irreducible separable permutations. The equations are used to find a third degree equation for joint distribution of ascents and descents on separable permutations that generalizes the respective known result for the descent distribution. Moreover, our general functional equations allow us to derive explicitly (joint) distribution of any subset of maxima and minima statistics on irreducible, reducible and all separable permutations. In particular, there are two equivalence classes of distributions of a pair of maxima or minima statistics. Finally, we present three unimodality conjectures about distributions of statistics on separable permutations.

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1 Introduction

A permutation of length n is a rearrangement of the set $[n] := \{1, 2, \ldots, n\}$. Denote by S_n the set of permutations of $[n]$ and let ε be the empty permutation. A *pattern* is a permutation. A permutation $\pi_1 \pi_2 \cdots \pi_n \in S_n$ avoids a pattern $p = p_1 p_2 \cdots p_k \in S_k$ if there is no subsequence $\pi_{i_1} \pi_{i_2} \cdots \pi_{i_k}$ such

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that $\pi_{i_j} < \pi_{i_m}$ if and only if $p_j < p_m$. Let $S_n(p)$ denote the set of p-avoiding permutations of length n. For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, its *reverse* is the permutation $r(\pi) = \pi_n \pi_{n-1} \cdots \pi_1$ and its *complement* is the permutation $c(\pi) = c(\pi_1)c(\pi_2)\cdots c(\pi_n)$ where $c(x) = n + 1 - x$. Also, the length of a permutation π, denoted by $|\pi|$, is the number of elements in π. For example, for $\pi = 423165$, $|\pi| = 6$, $r(\pi) = 561324$, and $c(\pi) = 354612$.

For $n \geq 2$, a permutation $\pi_1 \pi_2 \cdots \pi_n$ is *irreducible* if there is no i, $2 \leq i \leq n$, such that any element in $\pi_1 \cdots \pi_{i-1}$ is less than each element in $\pi_i \cdots \pi_n$. For example, the permutation 42513 is irreducible. By definition, the permutation 1 is irreducible. A permutation is *reducible* if it is not irreducible. The empty permutation ε is neither reducible nor irreducible. Each reducible permutation can be cut into irreducible pieces called *irreducible components*, or just *components*. For example, the components of the permutation 312546978 are 312, 54, 6, and 978.

1.1 Separable permutations

Suppose $\pi = \pi_1 \pi_2 \cdots \pi_m \in S_m$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$. We define the *direct sum* (or simply, *sum*) $\oplus$, and the *skew sum* $\ominus$ by building the permutations $\pi \oplus \sigma$ and $\pi \ominus \sigma$ as follows:

$$(\pi \oplus \sigma)_i = \begin{cases} \pi_i & \text{if } 1 \leq i \leq m, \\ \sigma_{i-m} + m & \text{if } m + 1 \leq i \leq m + n, \end{cases}$$

$$(\pi \ominus \sigma)_i = \begin{cases} \pi_i + n & \text{if } 1 \leq i \leq m, \\ \sigma_{i-m} & \text{if } m + 1 \leq i \leq m + n. \end{cases}$$

For example, $14325 \oplus 4231 = 143259786$ and $14325 \ominus 4231 = 587694231$.

The *separable permutations* are those which can be built from the permutation 1 by repeatedly applying the $\oplus$ and $\ominus$ operations. Bose, Buss and Lubiw [4] introduced the notion of separable permutation in 1998, and it is well-known [11] that the set of all separable permutations of length $n \geq 1$ is precisely $S_n(2413, 3142)$ (we assume here that by definition, the empty permutation ε is not separable). Separable permutations appear in the literature in various contexts (see, for example, [1, 2, 3, 7, 8, 9, 10, 12, 15]). Cleary, the reverse, or complement, or (usual group theoretic) inverse, or any composition of these operations applied to a separable permutation gives a separable permutation.

It is not difficult to see that any $\pi \in S_n(2413, 3142)$ has the following structure (see also Figure 1 for a schematic representation, where a permutation is viewed as a diagram with a dot in position (i, π_i) for each element

π_i of π):
$$\pi = L_1 L_2 \cdots L_m n R_m R_{m-1} \cdots R_1$$

where

- for $1 \le i \le m$, L_i and R_i are non-empty ($\ne \varepsilon$), with possible exception for L_1 and R_m, separable permutations which are intervals in π (that is, consist of all elements in $\{a, a+1, \dots, b\}$ for some a and b);

- $L_1 < R_1 < L_2 < R_2 < \cdots < L_m < R_m$, where $A < B$, for two permutations A and B, means that each element of A is less than every element of B. In particular, L_1, if it is not empty, contains 1.

For example, if $\pi = 2165743$ then $L_1 = 21$, $L_2 = 65$, $R_1 = 43$ and $R_2 = \varepsilon$. Note that a permutation in question is irreducible if and only if $L_1 = \varepsilon$.
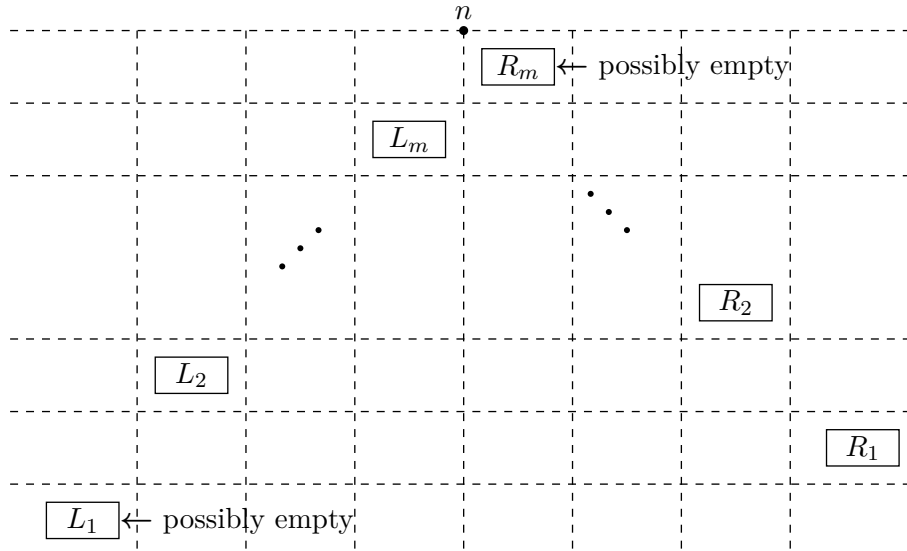

Figure 1: A schematic view of the permutation diagrams corresponding to separable permutations. Each L_i and R_j is a separable permutation.

It was conjectured by Shapiro and Getu and, for the first time, proved by West [17] that $S_n(3142, 2413)$ is counted by the $(n-1)$-th *Schröder number* (this is the sequence [14, A006318] beginning with 1, 2, 6, 22, 90, 394, 1806,...). The proof involves studying the generating tree for the restricted

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permutations and it uses a well-known relation between the Schröder numbers s_n and the Catalan numbers C_n:

$$s_n = \sum_{i=0}^{n} \binom{2n-i}{i} C_{n-i}. \tag{1}$$

Several other proofs of the enumeration formula for $S_n(3142, 2413)$ appear in the literature (see [11, Sec 2.2.5]), and an alternative to formula (1) was used in [13]:

$$s_n = \sum_{k=0}^{n} 2^k C_{n,k}, \tag{2}$$

where $C_{n,k}$ is the number of Dyck paths of length $2n$ with k peaks. The generating function (g.f.) for separable permutations is

$$S(t) := \sum_{n \geq 1} |S_n(3142, 2413)| t^n = \frac{1 - t - \sqrt{1 - 6t + t^2}}{2}. \tag{3}$$

In our studies, we also need the g.f. $I(t)$ for irreducible separable permutations. Note that for $n \geq 2$, the number of irreducible separable permutations in S_n is the same as the number of reducible separable permutations, which can be seen by applying the reverse operation to all separable permutations (sending reducible permutations to irreducible, and vice versa). Hence,

$$I(t) = t + \frac{1}{2}(S(t) - t) = \frac{1 + t - \sqrt{1 - 6t + t^2}}{4}. \tag{4}$$

The sequence of irreducible separable permutations begins with 1, 1, 3, 11, 45, 197, 903,... and these are the little Schröder numbers [14, A001003].

1.2 Distributions of permutation statistics

For a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$, the *descent* (resp., *ascent*) *statistic* on π, $\mathrm{des}(\pi)$ (resp., $\mathrm{asc}(\pi)$), is defined as the number of $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$ (resp., $\pi_i < \pi_{i+1}$). For example, $\mathrm{des}(561423) = 2$ and $\mathrm{asc}(25134) = 3$. The distribution of descents (or ascents) over S_n is given by the *Eulerian polynomial*

$$A_n(q) := \sum_{\pi \in S_n} q^{\mathrm{des}(\pi)} = \sum_{k=1}^{n} k! S(n, k)(q - 1)^{n-k}$$

where $S(n, k)$ is the *Stirling number of the second kind*.

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The distribution of descents has been studied over various subsets of permutations in the literature. For example, the distribution of descents over $S_n(231)$ is given by the *Narayana numbers*

$$\frac{1}{n}\binom{n}{k}\binom{n}{k+1}$$

(see [16]). Also, the distribution of descents over 123-avoiding permutations is given by the following formula [5, 6], where t and q correspond to the length and the number of descents:

$$\frac{-1 + 2tq + 2t^2q - 2tq^2 - 4t^2q^2 + 2t^2q^3 + \sqrt{1 - 4tq - 4t^2q + 4t^2q^2}}{2tq^2(tq - 1 - t)}.$$

The number of 321-avoiding permutations of length n with k descents [14, A091156] is given by

$$\frac{1}{n+1}\binom{n+1}{k}\sum_{j=0}^{n-2k}\binom{k+j-1}{k-1}\binom{n+1-k}{n-2k-j}.$$

More relevant to our paper is the result in [7, Cor 2.4] stating that the g.f. $S(t, q)$ giving the distribution of descents (equivalently, ascents, because of the reverse operation) on separable permutations satisfies

$$qS^3(t, q) + qtS^2(t, q) + ((1 + q)t - 1)S(t, q) + t = 0. \tag{5}$$

For a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, π_i is a *right-to-left maximum* (resp., *right-to-left minimum*) if $\pi_i > \pi_j$ (resp., $\pi_i < \pi_j$) for all $j > i$. In particular, π_n is a right-to-left maximum and a right-to-left minimum. The number of right-to-left maxima (resp., right-to-left minima) is denoted by $\mathrm{rmax}(\pi)$ (resp., $\mathrm{rmin}(\pi)$). For example, $\mathrm{rmax}(426513) = 3$ and $\mathrm{rmin}(426153) = 2$. Also, π_i is a *left-to-right maximum* (resp., *left-to-right minimum*), denoted by $\mathrm{lmax}(\pi)$ (resp., $\mathrm{lmin}(\pi)$) if $\pi_i > \pi_j$ (resp., $\pi_i < \pi_j$) for all $j < i$. In particular, π_1 is a left-to-right maximum and a left-to-right minimum. For example, $\mathrm{lmax}(425163) = \mathrm{lmin}(426153) = 3$.

Note that the distribution of rmax, which is the same as the distribution of rmin, or lmax, or lmin, or cycles in (unrestricted) permutations, is given by the *signless Stirling numbers of the first kind*. It is easy to see, and is well-known that

$$\sum_{\pi \in S_n} y^{\mathrm{rmax}(\pi)} = y(y + 1) \cdots (y + n - 1) = y^{(n)}$$

is given by the *rising factorial*.

Two k-tuples of (permutation) statistics $(s_1, s_2, \ldots, s_k)$ and $(s'_1, s'_2, \ldots, s'_k)$ are *equidistributed* over a set S if

$$\sum_{a \in S} t^{|a|} t_1^{s_1(a)} t_2^{s_2(a)} \cdots t_k^{s_k(a)} = \sum_{a \in S} t^{|a|} t_1^{s'_1(a)} t_2^{s'_2(a)} \cdots t_k^{s'_k(a)}$$

where $|a|$ is the size of a (for permutations, the size is the length).

In this paper, we are interested in the following g.f.

$$S(t, p, q, x, y, u, v) := \sum_{\pi} t^{|\pi|} p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)} x^{\mathrm{lmax}(\pi)} y^{\mathrm{rmax}(\pi)} u^{\mathrm{lmin}(\pi)} v^{\mathrm{rmin}(\pi)}$$

where the sum is taken over all separable permutations, and $I(t, p, q, x, y, u, v)$ is the respective g.f. on irreducible separable permutations. For brevity, throughout the paper we often omit the variables in the arguments that are set to be 1. For example, $S(t, p, q, 1, y, 1, 1)$ and $I(t, p, q, 1, 1, 1, 1)$ can be denoted by us by $S(t, p, q, y)$ and $I(t, p, q)$, respectively. This will not cause any confusion.

There are 6 unordered pairs and 4 unordered triples of statistics in $\{\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}, \mathrm{rmin}\}$. A permutation of statistics in a k-tuple of statistics results in a k-tuple of statistics having the same distribution modulo renaming variables, and we need to consider only one pair or triple from each such equivalence class. The following lemma shows that with respect to distributions, there is one equivalence class for a single statistic, one equivalence class for triples of statistics, and two equivalence classes for pairs of statistics.

Lemma 1. *The (pairs, tripples of) statistics in each of the sets*

$$\{\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}, \mathrm{rmin}\} \tag{6}$$

$$\{(\mathrm{lmax}, \mathrm{rmax}), (\mathrm{lmin}, \mathrm{rmin}), (\mathrm{lmin}, \mathrm{lmax}), (\mathrm{rmin}, \mathrm{rmax})\} \tag{7}$$

$$\{(\mathrm{rmax}, \mathrm{lmin}), (\mathrm{lmax}, \mathrm{rmin})\} \tag{8}$$

$$\{(\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}), (\mathrm{lmin}, \mathrm{rmin}, \mathrm{lmax}), (\mathrm{rmin}, \mathrm{rmax}, \mathrm{lmin}), \tag{9}$$
$$(\mathrm{rmax}, \mathrm{rmin}, \mathrm{lmax})\}$$

are equidistributed on (separable) permutations.

Proof. Applying the reverse and complement operations and their composition to all (separable) permutations, we obtain equidistribution of the single and triple statistics, as well as the pairs in set (8). On the other hand, it

is not difficult to see from the definitions that under the inverse operation, the statistic rmax goes to the statistic rmax, while the statistic lmax goes to the statistic rmin proving equidistribution of (lmax, rmax) and (rmin, rmax). The remaining equidistributions in set (7) are obtained by keeping track of transformations of the statistics in (lmax, rmax) and (rmin, rmax) under the reverse and complement operations. □

Remark 2. *By Lemma 1, to determine (joint) distribution of at most three statistics in the set {lmax, rmax, lmin, rmin}, it is sufficient to find the distribution of rmax and the joint distributions of the pairs of statistics (lmax, rmax) and (rmax, lmin) and the triple of statistics (lmax, rmax, lmin). Also, by the definitions, if π is a permutation of length at least 2 then π is irreducible if and only if*

- *inverse of π is irreducible;*

- *reverse of π is reducible;*

- *complement of π is reducible.*

1.3 Our results in this paper

Firstly, in Section 2 we will prove the following theorem.

Theorem 3. *The g.f. $S(t, p, q, x, y, u, v)$ and $I(t, p, q, x, y, u, v)$ satisfy functional equations (14) and (15).*

As an immediate corollary of Theorem 3, by setting all variables but t to be 1, we obtain the g.f.'s (3) and (4). As another corollary of Theorem 3 (the case of $x = y = u = v = 1$) we will obtain the following theorem generalizing the known relation (5) for the distribution of descents on separable permutations (substituting $p = 1$ in (10) gives (5)).

Theorem 4. *The g.f. $S(t, p, q)$ and $I(t, p, q)$ giving the joint distribution of ascents and descents on (irreducible) separable permutations satisfy*

$$pq(S(t, p, q))^3 + pq(S(t, p, q))^2 t + S(t, p, q)((p + q)t - 1) + t = 0, \quad (10)$$

$$I(t, p, q) = \frac{t + q(t + S(t, p, q))S(t, p, q)}{1 + qS(t, p, q)}. \quad (11)$$

Furthermore, letting all variables in Theorem 3, but t and y, be equal to 1, then replacing y by z and applying Remark 2, we will justify the following theorem.

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g.f.	irreducible	reducible
$\frac{S^2(t,z)}{1+S(t,z)} + zt$	stat $\in \{\text{rmax}, \text{lmin}\}$	stat $\in \{\text{lmax}, \text{rmin}\}$
$\frac{S(t,z)}{1+S(t,z)} - zt$	stat $\in \{\text{lmax}, \text{rmin}\}$	stat $\in \{\text{rmax}, \text{lmin}\}$

Table 1: Distributions of statistics in Theorem 5

Theorem 5. *The g.f. $S(t,z) := \sum_\pi t^{|\pi|} z^{\text{stat}(\pi)}$ giving the distribution of any statistic* stat $\in \{\text{lmax}, \text{rmax}, \text{lmin}, \text{rmin}\}$ *on separable permutations is*

$$\frac{4\sqrt{\left(-\frac{1}{4}\sqrt{t^2 - 6t + 1} - tz + \frac{t}{4} + \frac{5}{4}\right)^2 + \sqrt{t^2 - 6t + 1} - t - 1} - \sqrt{t^2 - 6t + 1} + 4tz + t - 3}{2\left(\sqrt{t^2 - 6t + 1} - t - 1\right)}.$$

The distributions of stat *on irreducible and reducible separable permutations are given in Table 1. (Tables 3, 4, 5 give initial values for the distributions.)*

As for joint equidistributions, we will prove the following four theorems.

Theorem 6. *The g.f. $S(t, z_1, z_2) := \sum_\pi t^{|\pi|} z_1^{\text{stat}_1(\pi)} z_2^{\text{stat}_2(\pi)}$ giving the distribution of* $(\text{stat}_1, \text{stat}_2)$ *in set (7) on separable permutations is*

$$\frac{(S(t, z_1) + 1)(S(t, z_2) + 1)tz_1 z_2}{1 - S(t, z_1)S(t, z_2)}$$

where $S(t,z)$ is given in Theorem 5. The distributions of

$$(\text{stat}_1, \text{stat}_2) \in \{(\text{lmax}, \text{rmax}), (\text{lmax}, \text{lmin}), (\text{rmin}, \text{rmax}), (\text{rmin}, \text{lmin})\}$$

on irreducible and reducible separable permutations are given by $\frac{(S(t, z_2)+1)tz_1 z_2}{1 - S(t,z_1)S(t,z_2)}$ and $\frac{S(t,z_1)(S(t,z_2)+1)tz_1 z_2}{1 - S(t,z_1)S(t,z_2)}$, respectively.

Theorem 7. *The g.f. $S(t, z_1, z_2) := \sum_\pi t^{|\pi|} z_1^{\text{stat}_1(\pi)} z_2^{\text{stat}_2(\pi)}$ giving the distribution of* $(\text{stat}_1, \text{stat}_2)$ *in set (8) on separable permutations is*

$$\frac{S(t, z_1)I(t, z_2) + tz_1 z_2}{1 - S(t, z_1)I(t, z_2) - tz_1 z_2}$$

where $S(t, z_1)$ and $I(t, z_2)$ are given in Theorem 5. The distributions of $(\text{stat}_1, \text{stat}_2)$ on irreducible and reducible separable permutations are recorded in Table 2.

In Theorems 8 and 9 we use the following function:

$$E(t, z_1, z_2, z_3) := z_1 z_2 z_3 t + \frac{(S(t,z_1)+1)(S(t,z_2)+1)(S(t,z_3)+1)t^2 z_1^2 z_2 z_3}{(1 - S(t,z_1)S(t,z_3))(1 - S(t,z_1)S(t,z_2))}. \quad (12)$$

Theorem 8. *The g.f.* $S(t, z_1, z_2, z_3) := \sum_\pi t^{|\pi|} z_1^{\mathrm{stat}_1(\pi)} z_2^{\mathrm{stat}_2(\pi)} z_3^{\mathrm{stat}_3(\pi)}$ *giving the distribution of* $(\mathrm{stat}_1, \mathrm{stat}_2, \mathrm{stat}_3)$ *in set* (9) *on separable permutations is*

$$\frac{E(t, z_1, z_2, z_3)}{1 - A(z_2, z_3) - t z_2 z_3}$$

where $A(z_2, z_3) = S(t, z_2)\left(z_3 t + \frac{S^2(t,z_3)}{S(t,z_3)+1}\right)$ *and* $S(t, z)$ *is given in Theorem 5. The distributions of* $(\mathrm{stat}_1, \mathrm{stat}_2, \mathrm{stat}_3)$ *in set* (9) *on irreducible separable permutations (denoted by* $I(t, z_1, z_2, z_3)$*) and reducible separable permutations is given by the following two cases:*

- *The distribution of* $(\mathrm{stat}_1, \mathrm{stat}_2, \mathrm{stat}_3)$ *in*

$$\{(\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}), (\mathrm{rmin}, \mathrm{rmax}, \mathrm{lmin})\}$$

  on reducible separable permutations is given by

$$E(t, z_1, z_2, z_3) - z_1 z_2 z_3 t \quad (13)$$

  and on irreducible separable permutations by

$$\frac{1}{1 - A(z_2, z_3) - t z_2 z_3}\left(xyut + (A(z_2, z_3) + t z_2 z_3)(E(t, z_1, z_2, z_3) - z_1 z_2 z_3 t)\right).$$

- *The distribution of* $(\mathrm{stat}_1, \mathrm{stat}_2, \mathrm{stat}_3)$ *in*

$$\{(\mathrm{lmin}, \mathrm{rmin}, \mathrm{lmax}), (\mathrm{rmax}, \mathrm{rmin}, \mathrm{lmax})\}$$

  on irreducible separable permutations is given by $E(t, z_1, z_2, z_3)$ *and on reducible separable permutations by*

$$-z_1 z_2 z_3 t + \frac{1}{1 - A(z_2, z_3) - t z_2 z_3}(xyut + (A(z_2, z_3) + t z_2 z_3)(E(t, z_1, z_2, z_3) - z_1 z_2 z_3 t)).$$

Theorem 9. *The g.f.* $S(t, x, y, u, v) := \sum_\pi t^{|\pi|} x^{\mathrm{lmax}(\pi)} y^{\mathrm{rmax}(\pi)} u^{\mathrm{lmin}(\pi)} v^{\mathrm{rmin}(\pi)}$ *giving the distribution of the four statistics over separable permutations is*

$$xyuvt + E(t, x, y, u)S(t, y, u, v) + E(t, x, y, v)S(t, x, u, v)$$

where $S(t, y, u, v)$ *and* $S(t, x, u, v)$ *are given by Theorem 8 and* $E(t, z_1, z_2, z_3)$ *by* (12)*. The respective g.f.* $I(t, x, y, u, v)$ *for irreducible separable permutations is*

$$xyuvt + E(t, x, y, u)S(t, y, u, v)$$

and for reducible separable permutations is $E(t, x, y, v)S(t, x, u, v)$.

g.f.	irreducible	reducible
$\frac{S(t,z_1)I(t,z_2)+tz_1z_2}{1-S(t,z_1)I(t,z_2)-tz_1z_2} - S(t,z_1)I(t,z_2)$	$(\mathrm{rmax},\mathrm{lmin})$	$(\mathrm{lmax},\mathrm{rmin})$
$S(t,z_1)I(t,z_2)$	$(\mathrm{lmax},\mathrm{rmin})$	$(\mathrm{rmax},\mathrm{lmin})$

Table 2: Distributions of statistics in Theorem 7 depending on $(\mathrm{stat}_1,\mathrm{stat}_2)$

Finally, in Section 3 we give several unimodality conjectures confirming which would extend our knowledge in [7] on unimodality of the distribution of descents on separable permutations.

2   The key functional equations and specializations

We begin with deriving the following functional equations:

$$S(t,p,q,x,y,u,v) = xyuvt + pS(t,p,q,x,1,u,v)I(t,p,q,x,y,1,v) + \quad (14)$$
$$q(S(t,p,q,x,y,u,1) - I(t,p,q,x,y,u,1) + xyut)S(t,p,q,1,y,u,v);$$
$$I(t,p,q,x,y,u,v) = xyuvt + \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (15)$$
$$q(S(t,p,q,x,y,u,1) - I(t,p,q,x,y,u,1) + xyut)S(t,p,q,1,y,u,v).$$

Referring to the structure of a separable permutation π in Figure 1, we have the following four disjoint possibilities, where Cases 1–3 cover irreducible permutations and Case 4 covers reducible permutations.

1. $\pi = 1$ giving the term of $xyuvt$.

2. $\pi = nR_1 = 1 \ominus R_1$, $R_1 \neq \varepsilon$, giving the term of $qxyutS(t,p,q,1,y,u,v)$ because $\mathrm{lmax}(R_1)$ does not contribute to $\mathrm{lmax}(\pi) = 1$ and R_1 can be any separable permutation, while n contributes an extra descent and one extra left-to-right minimum and one extra right-to-left maximum.

3. $\pi = \pi' \ominus R_1$ where $R_1 \neq \varepsilon$ $(L_1 = \varepsilon)$ and π' is a reducible separable permutation (of length at least 2). In this case, we have the term of

    $$q(S(t,p,q,x,y,u,1) - I(t,p,q,x,y,u,1))S(t,p,q,1,y,u,v)$$

    since

    - π' does not contribute to the statistic rmin in π and hence has g.f. $S(t,p,q,x,y,u,1) - I(t,p,q,x,y,u,1)$,

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- R_1 does not contribute to the statistic lmax in π and hence has g.f. $S(t, p, q, 1, y, u, v)$,

- π' is independent from R_1, and

- an extra descent is formed between π' and R_1.

4. $\pi = L_1 \oplus \pi''$ where $L_1 \neq \varepsilon$ and π'' is an irreducible separable permutation. In this case, we have the term of $pS(t, p, q, x, 1, u, v)I(t, p, q, x, y, 1, v)$ since

  - L_1 does not contribute to the statistic rmax in π,

  - π'' does not contribute to the statistic lmin in π,

  - L_1 is independent from π'', and

  - an extra ascent is formed between L_1 and π''.

Summarising Cases 1–4 gives (14). Summarising Cases 1–3 gives (15). Hence, Theorem 3 holds.

The following theorem will be useful for us to rewrite, in several occasions, special cases coming from (14) (after setting some of the variables be equal to 1) in a shorter form.

Theorem 10. *The following equalities hold*

$$
\begin{aligned}
S(t, p, q, x, y, u, v) - I(t, p, q, x, y, u, v) &= pI(t, p, q, x, u, v)S(t, p, q, x, y, v) \\
&= pI(t, p, q, x, y, v)S(t, p, q, x, u, v). \quad (16)
\end{aligned}
$$

Proof. Clearly, $S(t, p, q, x, y, u, v) - I(t, p, q, x, y, u, v)$ is the g.f. for all reducible separable permutations. We have two more ways to generate all such permutations. Indeed, each reducible permutation π is of the form $\pi' \oplus \pi''$, and we can think of π' and π'' in two ways obtaining the equalities:

- π' is an irreducible separable permutation and π'' is any separable permutation;

- π' is any separable permutation and π'' is an irreducible separable permutation.

In both cases $\mathrm{rmax}(\pi')$ (resp., $\mathrm{lmin}(\pi'')$) does not contribute to $\mathrm{rmax}(\pi)$ (resp., $\mathrm{lmin}(\pi)$) and hence the variable y (resp., u) must be set to 1 in the respective g.f.'s; also, the factor of q corresponds to the extra ascent formed between π' and π''. □

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The following theorem is a direct consequence to Remark 2 and the fact that the permutation 1 is irreducible, and it is used by us implicitly when considering a single set of statistics from an equivalence class in proofs of theorems.

Theorem 11. *Suppose that I and R are g.f.'s giving joint distributions of statistics in $(\mathrm{stat}_1, \ldots, \mathrm{stat}_m)$ recorded, respectively, by variables in $\{z_1, \ldots, z_m\}$ on irreducible and reducible separable permutations, respectively (and t records the length of permutations). Further, let statistic stat_i goes to statistic stat'_i for $1 \leq i \leq m$ under the reverse (or complement) operation. Then $I - z_1 \cdots z_m t$ and $R + z_1 \cdots z_m t$ are g.f. for joint distributions of statistics in $(\mathrm{stat}'_1, \ldots, \mathrm{stat}'_m)$ on reducible and irreducible separable permutations, respectively.*

2.1 Joint distribution of ascents and descents

Our next goal is proving Theorem 4 for

$$S(t, p, q) = \sum_{\pi} t^{|\pi|} p^{\mathrm{asc}(\pi)} q^{\mathrm{des}(\pi)}$$

where the sum is taken over all separable permutations, and for $I(t, p, q)$ obtained by restricting $S(t, p, q)$ to irreducible permutations.

By (16), we have

$$S(t, p, q) - I(t, p, q) = pI(t, p, q)S(t, p, q). \tag{17}$$

Setting $x = y = u = v = 1$ in (14) and (15), and using (17) in the specialization of (14), we obtain

$$S(t, p, q) = t(qS(t, p, q) + 1) + pqIS^2(t, p, q) + pI(t, p, q)S(t, p, q); \tag{18}$$

$$I(t, p, q) = t(qS(t, p, q) + 1) + q(S(t, p, q) - I)S(t, p, q). \tag{19}$$

Solving (19) for $I(t, p, q)$ we obtain (11), and substituting the solution into (18), we obtain (10) hence completing our proof of Theorem 4.

2.2 Distribution of a statistic in $\{\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}, \mathrm{rmin}\}$

By Remark 2, to prove Theorem 5 it is sufficient to prove it for the g.f.

$$S(t, y) = \sum_{\pi} t^{|\pi|} y^{\mathrm{rmax}(\pi)}$$

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where the sum is taken over all separable permutations, and for the respective g.f. $I(t, y)$ obtained by restricting $S(t, y)$ to irreducible permutations.

By (16), we have

$$S(t, y) - I(t, y) = I(t)S(t, y) = I(t, y)S(t). \tag{20}$$

Setting $p = q = x = u = v = 1$ in (14) and (15), and using (20) in the specialization of (14), we obtain

$$S(t, y) = yt(S(t, y) + 1) + I(t)(S(t, y) + 1)S(t, y); \tag{21}$$

$$I(t, y) = yt(S(t, y) + 1) + (S(t, y) - I(t, y))S(t, y). \tag{22}$$

Solving simultaneously the system of equations (21) and (22), and using the observations in Remark 2 on reducibility of permutations under reverse and complement, we complete the proof of Theorem 5.

In Tables 3, 4 and 5 we present, respectively, initial values for the distribution of a statistic in $\{\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}, \mathrm{rmin}\}$ on separable permutations, a statistic in $\{\mathrm{rmax}, \mathrm{lmin}\}$ on irreducible separable permutations, and a statistic in $\{\mathrm{lmax}, \mathrm{rmin}\}$ on irreducible separable permutations.

2.3 Joint distribution of lmax and rmax

By Remark 2, to prove Theorem 6 it is sufficient to prove it for the g.f.

$$S(t, x, y) = \sum_{\pi} t^{|\pi|} x^{\mathrm{lmax}(\pi)} y^{\mathrm{rmax}(\pi)}$$

where the sum is taken over all separable permutations, and for the respective g.f. $I(t, x, y)$ for irreducible separable permutations.

By (16), we have

$$S(t, x, y) - I(t, x, y) = S(t, x)I(t, x, y) = S(t, x, y)I(t, x). \tag{23}$$

Setting $p = q = u = v = 1$ in (14) and (15), and using (23) in the specialization of (14), we obtain

$$S(t, x, y) = xyt(S(t, y) + 1) + S(t, x, y)I(t, x)(S(t, y) + 1); \tag{24}$$

$$I(t, x, y) = xyt(S(t, y) + 1) + (S(t, x, y) - I(t, x, y))S(t, y). \tag{25}$$

Solving simultaneously the system of equations (24) and (25), and using the observations in Remark 2 on reducibility of permutations under reverse,

complement and inverse, and Theorem 11, we complete the proof of Theorem 6.

The initial terms of $S(t, x, y)$ are

$$t(xy) + t^2(x^2y + xy^2) + t^3(x^3y + 2x^2y^2 + xy^3 + x^2y + xy^2) +$$
$$t^4(x^4y + 3x^3y^2 + 3x^2y^3 + xy^4 + 3x^3y + 4x^2y^2 + 3xy^3 + 2x^2y + 2xy^2) + \cdots .$$

The initial terms of $I(t, x, y)$ are

$$t(xy) + t^2(xy^2) + t^3(x^2y^2 + xy^3 + xy^2) + t^4(x^3y^2 + 2x^2y^3 + xy^4 + 2x^2y^2 + 3xy^3 + 2xy^2) +$$
$$t^5(x^4y^2 + 3x^3y^3 + 3x^2y^4 + xy^5 + 4x^3y^2 + 7x^2y^3 + 6xy^4 + 5x^2y^2 + 9xy^3 + 6xy^2) + \cdots .$$

2.4 Joint distribution of rmax and lmin

By Remark 2, to prove Theorem 7 it is sufficient to prove it for the g.f.

$$S(t, y, u) = \sum_\pi t^{|\pi|} y^{\mathrm{rmax}(\pi)} u^{\mathrm{lmin}(\pi)}$$

where the sum is taken over all separable permutations, and for the respective g.f. $I(t, y, u)$ for irreducible separable permutations.

Setting $p = q = x = v = 1$ in (14) and (15) we obtain

$$S(t, y, u) = yut + S(t, u)I(t, y) + (S(t, y, u) - I(t, y, u) + yut)S(t, y, u); \quad (26)$$

$$I(t, y, u) = yut + (S(t, y, u) - I(t, y, u) + yut)S(t, y, u). \quad (27)$$

Solving simultaneously the system of equations (26) and (27), and using the observations in Remark 2 on reducibility of permutations under reverse and complement, and Theorem 11, we complete the proof of Theorem 7. Note that in Theorem 7, by (16), $S(t, u)I(t, y) = I(t, u)S(t, y)$.

The initial terms of $S(t, y, u)$ are

$$t(uy) + t^2(u^2y^2 + uy) + t^3(u^3y^3 + 2u^2y^2 + u^2y + uy^2 + uy) +$$
$$t^4(u^4y^4 + 3u^3y^3 + 2u^3y^2 + 2u^2y^3 + u^3y + 4u^2y^2 + uy^3 + 3u^2y + 3uy^2 + 2uy) + \cdots .$$

The initial terms of $I(t, y, u)$ are

$$t(uy) + t^2(u^2y^2) + t^3(u^3y^3 + 2u^2y^2) + t^4(u^4y^4 + 3u^3y^3 + 2u^3y^2 + 2u^2y^3 + 3u^2y^2) +$$
$$t^5(u^5y^5 + 4u^4y^4 + 3u^4y^3 + 3u^3y^4 + 2u^4y^2 + 8u^3y^3 + 2u^2y^4 + 8u^3y^2 + 8u^2y^3 + 6u^2y^2) + \cdots .$$

2.5 Joint distribution of lmax, rmax and lmin

By Remark 2, to prove Theorem 8 it is sufficient to prove it for the g.f.

$$S(t, x, y, u) = \sum_\pi t^{|\pi|} x^{\mathrm{lmax}(\pi)} y^{\mathrm{rmax}(\pi)} u^{\mathrm{lmin}(\pi)}$$

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k $\diagdown$ n	1	2	3	4	5	6	7	8
1	1							
2	1	1						
3	2	3	1					
4	6	9	6	1				
5	22	31	26	10	1			
6	90	120	108	60	15	1		
7	394	504	461	305	120	21	1	
8	1806	2240	2046	1475	745	217	28	1

Table 3: Distribution of stat $\in \{\mathrm{lmax}, \mathrm{rmax}, \mathrm{lmin}, \mathrm{rmin}\}$, where n is the length of permutations and k is the number of occurrences of stat.

where the sum is taken over all separable permutations, and for the respective g.f. $I(t, x, y, u)$ for irreducible separable permutations.

Setting $p = q = v = 1$ in (14) and (15) we obtain

$$
\begin{aligned}
S(t, x, y, u) &= xyut + S(t, x, u)I(t, x, y) + \\
&\quad (S(t, x, y, u) - I(t, x, y, u) + xyut)S(t, y, u);
\end{aligned}
\tag{28}
$$

$$
I(t, x, y, u) = xyut + (S(t, x, y, u) - I(t, x, y, u) + xyut)S(t, y, u).
\tag{29}
$$

The system of equations (28) and (29) is equivalent to the system

$$
S(t, x, y, u) = S(t, x, u)I(t, x, y) + I(t, x, y, u);
\tag{30}
$$

$$
I(t, x, y, u) = \frac{xyut + (S(t, x, y, u) + xyut)S(t, y, u)}{S(t, y, u) + 1}.
\tag{31}
$$

Observe that from (30), the distribution of the statistics over reducible permutations is $S(t, x, u)I(t, x, y)$. Solving simultaneously the system of equations (30) and (31), which involves using the formulas $S(t, x, u) = \frac{(S(t,x)+1)(S(t,u)+1)txu}{1-S(t,x)S(t,u)}$ and $I(t, x, y) = \frac{(S(t,y)+1)txy}{1-S(t,x)S(t,y)}$ from Theorem 6 and the formula $S(t, y, u) = \frac{S(t,y)I(t,u)+tyu}{1-S(t,y)I(t,u)-tyu}$ from Theorem 7, we complete the proof of Theorem 8.

The initial terms of $S(t, x, y, u)$ are

$t(uxy) + t^2(u^2xy^2 + ux^2y) +$
$t^3(u^3xy^3 + u^2x^2y^2 + u^2x^2y + ux^3y + u^2xy^2 + ux^2y^2) +$
$t^4(u^4xy^4 + u^3x^2y^3 + u^3x^2y^2 + u^2x^3y^2 + 2u^3xy^3 + u^2x^2y^3 + u^3x^2y + 2u^2x^3y + ux^4y +$
$u^3xy^2 + 2u^2x^2y^2 + 2ux^3y^2 + u^2xy^3 + ux^2y^3 + u^2x^2y + ux^3y + u^2xy^2 + ux^2y^2) + \cdots.$

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The initial terms of $I(t,x,y,u)$ are

$$t(uxy) + t^2(u^2xy^2) + t^3(u^3xy^3 + u^2x^2y^2 + u^2xy^2) + t^4(u^4xy^4 + u^3x^2y^3 +$$
$$u^3x^2y^2 + u^2x^3y^2 + 2u^3xy^3 + u^2x^2y^3 + u^3xy^2 + u^2x^2y^2 + u^2xy^3 + u^2xy^2) + \cdots.$$

2.6 Joint distribution of lmax, rmax, lmin and rmin

Setting $p = q = 1$ in (14) and (15), we obtain

$$
\begin{aligned}
S(t,x,y,u,v) &= xyuvt + S(t,x,u,v)I(t,x,y,v) + \qquad\qquad (32)\\
&\quad (S(t,x,y,u) - I(t,x,y,u) + xyut)S(t,y,u,v);\\
I(t,x,y,u,v) &= xyuvt +\\
&\quad (S(t,x,y,u) - I(t,x,y,u) + xyut)S(t,y,u,v). \quad (33)
\end{aligned}
$$

Subtracting from (32) the equation (33), we obtain

$$S(t,x,y,u,v) = I(t,x,y,u,v) + S(t,x,u,v)I(t,x,y,v)$$

and the statements in Theorem 9 follow after observing that $S(t,x,y,u) - I(t,x,y,u)$ in (33) records the joint distribution of the statistics lmax, rmax and lmin on reducible separable permutations and hence is given by (13).

The initial terms of $S(t,x,y,u,v)$ are

$$t(uvxy) + t^2(uv^2x^2y + u^2vxy^2) +$$
$$t^3(uv^3x^3y + u^3vxy^3 + u^2v^2x^2y + u^2v^2xy^2 + u^2vx^2y^2 + uv^2x^2y^2) +$$
$$t^4(uv^4x^4y + u^4vxy^4 + u^2v^3x^3y + uv^3x^3y^2 + u^3v^2xy^3 + u^3vx^2y^3 + u^3v^2x^2y +$$
$$u^2v^3x^2y + u^2v^2x^3y + uv^3x^3y + u^3v^2xy^2 + u^2v^3xy^2 + u^3vx^2y^2 + 2u^2v^2x^2y^2 +$$
$$uv^3x^2y^2 + u^2vx^3y^2 + uv^2x^3y^2 + u^3vxy^3 + u^2v^2xy^3 + u^2vx^2y^3 + uv^2x^2y^3) + \cdots.$$

The initial terms of $I(t,x,y,u,v)$ are

$$t(uvxy) + t^2(u^2vxy^2) + t^3(u^3vxy^3 + u^2v^2xy^2 + u^2vx^2y^2) +$$
$$t^4(u^4vxy^4 + u^3v^2xy^3 + u^3vx^2y^3 + u^3v^2xy^2 + u^2v^3xy^2 + u^3vx^2y^2 + u^2v^2x^2y^2 +$$
$$u^2vx^3y^2 + u^3vxy^3 + u^2v^2xy^3 + u^2vx^2y^3) + \cdots.$$

3 Unimodality conjectures

We conclude our paper with unimodality conjectures based on the computational results in Tables 3, 4 and 5. That would be interesting to find combinatorial proofs of the conjectures based on the appropriate embeddings, for example, of separable permutations with $k+1$ right-to-left maxima into separable permutations of the same length with k right-to-left maxima for $k \geq 2$ and separable permutations ending with the largest element into separable permutations of the same length with two right-to-left maxima.

k n	1	2	3	4	5	6	7	8
1	1							
2	0	1						
3	0	2	1					
4	0	5	5	1				
5	0	16	19	9	1			
6	0	60	73	49	14	1		
7	0	248	298	232	104	20	1	
8	0	1092	1288	1069	607	195	27	1

Table 4: Distribution of rmax or lmin (resp., lmax or rmin) on *irreducible* (resp., *reducible*) separable permutations, where n is the length of permutations and k is the number of occurrences of the statistic.

Conjecture 1. *The distribution of separable permutations $\pi \in S_n$ with* $\text{stat}(\pi) = k$, *stat* $\in \{\text{lmax}, \text{rmax}, \text{lmin}, \text{rmin}\}$, *is unimodal with the peak being at $k = 2$ for $n \geq 3$ (see Table 3).*

Conjecture 2. *The distribution of* rmax *or* lmin *(resp.,* lmax *or* rmin*) on* irreducible *(resp.,* reducible*) separable permutations is unimodal with the peak being at $k = 3$ for $n \geq 5$ (see Table 4).*

Conjecture 3. *The distribution of* rmax *or* lmin *(resp.,* lmax *or* rmin*) on* reducible *(resp.,* irreducible*) separable permutations is unimodal with the peak being at $k = 1$ for $n \geq 1$ (see Table 5).*

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n \ k	1	2	3	4	5	6	7
1	1						
2	1						
3	2	1					
4	6	4	1				
5	22	15	7	1			
6	90	60	35	11	1		
7	394	256	163	73	16	1	
8	1806	1148	758	406	138	22	1

Table 5: Distribution of rmax or lmin (resp., lmax or rmin) on *reducible* (resp., *irreducible*) separable permutations, where n is the length of permutations and k is the number of occurrences of the statistic.

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