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The log-concavity of Kazhdan-Lusztig polynomials of uniform matroids*

XIE Matthew H. Y. · ZHANG Philip B.

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Abstract Elias, Proudfoot, and Wakefield conjectured that the Kazhdan-Lusztig polynomial of any matroid is log-concave. Inspired by a computer proof of Moll's log-concavity conjecture given by Kauers and Paule, we use a computer algebra system to prove the conjecture for arbitrary uniform matroids.

Keywords Log-concavity, Kazhdan-Lusztig polynomial, uniform matroid, HolonomicFunctions.

1 Introduction

Elias, Proudfoot, and Wakefield [5] introduced the notion of the Kazhdan-Lusztig polynomials of matroids. The conjectures related to Kazhdan-Lusztig polynomials of matroids attracted the attention of algebraic geometers and combinatorialists.

Recently, Braden, Huh, Matherne, Proudfoot, and Wang [1] confirmed a conjecture of Elias, Proudfoot, and Wakefield [5], which states that the Kazhdan-Lusztig polynomial of an arbitrary matroid has only non-negative coefficients. Unlike the case of Coxeter groups, Elias, Proudfoot, and Wakefield also conjectured the log-concavity of these polynomials which is still open. Recall that a real polynomial $\sum_{i=0}^{n} a_i x^i$ is said to be *log-concave* if its coefficients satisfy that $a_i^2 \geq a_{i-1}a_{i+1}$ for any $1 \leq i \leq n-1$.

Conjecture 1 ([5]) For any matroid M, the Kazhdan-Lusztig polynomial $P_M(t)$ is log-concave.

Conjecture 1 has been confirmed for whirl matroids, wheel matroids, and graphic matroids of cycle graphs, fan graphs, and squares of paths, see [9, 13], where the real-rootedness of these polynomials has been proved, which implies their log-concavity by Newton's inequalities.

XIE Matthew H. Y.
College of Science, Tianjin University of Technology, Tianjin 300384, P. R. China.
ZHANG Philip B. (corresponding author)
College of Mathematical Science, Tianjin Normal University, Tianjin 300387, P. R. China.
Email: zhang@tjnu.edu.cn

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In this paper, we shall prove this conjecture for uniform matroids. Throughout this paper, we always assume that m and d are positive integers. Let $U_{m,d}$ denote the uniform matroid of rank d on a set of $m+d$ elements. Suppose that

$$P_{U_{m,d}}(t) = \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} c_{m,d,i} t^i.$$

Several different formulae for $c_{m,d,i}$ have been given, see [6–8, 12]. In this paper we use the following expression.

Theorem 1 ([7, Theorem 1.3]) *For any positive integers m, d and any nonnegative integer $i \le \lfloor \frac{d-1}{2} \rfloor$, we have*

$$c_{m,d,i} = \frac{1}{d-i}\binom{d+m}{i}\sum_{h=0}^{m-1}\binom{d-i+h}{h+i+1}\binom{i-1+h}{h}. \tag{1.1}$$

The real-rootedness of the Kazhdan-Lusztig polynomials of general uniform matroids is still open. Gedeon, Proudfoot, and Young [9] proved that the polynomial $P_{U_{1,d}}(t)$ has only negative zeros. By using a computer algebra system, Gao, Lu, Xie, Yang, and Zhang [7] proved that the polynomial $P_{U_{m,d}}(t)$ has only negative zeros for $2 \le m \le 15$. It is worth mentioning that the computer algebra system is also used to prove real-rootedness of other combinatorial polynomials, see [2, 4].

The main result of this paper is as follows.

Theorem 2 *For any positive integers m and d, the polynomial $P_{U_{m,d}}(t)$ is log-concave.*

The outline of our proof of Theorem 2 is as follows. For any positive integer i, we define

$$a_{n,i} = \frac{1}{(i-1)!}\sum_{h=0}^{m-1}\frac{1}{(h+i)(h+i+1)}\binom{h+n}{h},$$

and we let $a_{n,0} = 1$. For any nonnegative integer i, we define

$$b_{d,i} = \frac{(d+m)!}{(d-i+m)!}\binom{d-i-1}{i}.$$

Note that we ignore the index m in the subscripts of $a_{n,i}$ and $b_{d,i}$ for convenience throughout the paper. It is easy to check that

$$c_{m,d,i} = a_{d-i,i} b_{d,i}.$$

We divide the proof into two inequalities and prove them respectively.

Lemma 3 *For any positive integers m, i and $d \ge 2i+3$, we have*

$$a_{d-i,i}^2 \ge a_{d-i-1,i+1} a_{d-i+1,i-1}.$$

Let $n = d - i$. An equivalent statement of Lemma 3 is as follows: For any positive integers m, n and $i \le n-3$, we have

$$a_{n,i}^2 \ge a_{n-1,i+1} a_{n+1,i-1}. \tag{1.2}$$

Lemma 4 *For any positive integers m, i and $d \geq 2i+3$, we have*

$$b_{d,i}^2 \geq b_{d,i+1} b_{d,i-1}.$$

Our proof of Lemma 3 follows the idea of Kauers and Paule's computer proof [10] of Moll's log-concavity conjecture. We first present some recurrence relations of $a_{n,i}$ in Section 2 and then estimate upper and lower bounds of $\frac{a_{n,i}}{a_{n-1,i}}$ in Section 3. In Section 4, to prove Lemma 3, we divide (1.2) into three inequalities and prove them using a computer algebra system. Finally, we verify Lemma 4 directly and complete the proof of Theorem 2 in Section 5.

2 Recurrence relations of $a_{n,i}$

In this section, some recurrence relations of $a_{n,i}$, which will be used in later sections, are given. The recurrence relations are obtained from the **HolonomicFunctions** package[†] by Koutschan [11] for **Mathematica**. For more information of the method for finding recurrences of combinatorial sequences, we refer the reader to Chen and Kauers [3].

The **HolonomicFunctions** package can be imported into **Mathematica** in the following way.[‡]

```
In[1]:= << RISC`HolonomicFunctions`
        HolonomicFunctions Package version 1.7.3 (21-Mar-2017)
        written by Christoph Koutschan
        Copyright Research Institute for Symbolic Computation (RISC),
        Johannes Kepler University, Linz, Austria
```

The main result of this section is stated as follows.

Lemma 5 *For any positive integers m, i and $n \geq i+3$, we have*

$$a_{n+1,i} = \frac{2n^2 + (m-i)n - (m+i)}{n(n+1)} a_{n,i} - \frac{(n-i-1)(m+n)}{n(n+1)} a_{n-1,i}, \tag{2.1}$$

$$a_{n-1,i+1} = \frac{n(n-1)}{i(i+m+1)(n-i-2)} a_{n,i} - \frac{n^2 - (i+1)n + i(i+m+1)}{i(i+m+1)(n-i-2)} a_{n-1,i}, \tag{2.2}$$

$$a_{n-1,i-1} = \frac{(n-1)n}{i+m-1} a_{n,i} - \frac{(n-i-1)(i+m+n-1)}{i+m-1} a_{n-1,i}, \tag{2.3}$$

$$a_{n,i+1} = \frac{i+m+n}{i(i+m+1)} a_{n,i} - \frac{m+n}{i(i+m+1)} a_{n-1,i}. \tag{2.4}$$

Proof We first prove (2.1) and (2.2). The command **Annihilator**[*expr*] computes annihilating operators for the expression *expr*.

[†]The HolonomicFunctions package can be downloaded at https://www3.risc.jku.at/research/combinat/software/ergosum/RISC/HolonomicFunctions.html.

[‡] The source code of our computer programs presented in this paper is openly available in GitHub at https://github.com/mathxie/uniform_log-concave.

In[2]:= $\mathbf{ann = Annihilator\left[Sum\left[\frac{Binomial[n+h,h]}{(i-1)!(h+i)(h+i+1)}, \{h,0,m-1\}\right], \{S[i], S[n]\}\right]}$

Out[2]= $\{-i(1+i+m)(1+i-n)S_i - n(1+n)S_n + (i^2+im+n-in+n^2), (2+3n+n^2)S_n^2 + (-2-(4+m)n-2n^2+i(2+n))S_n - (i-n)(1+m+n)\}$

Here S_n (respectively S_i) denotes the forward shift in n (respectively i). We next use the command **ApplyOreOperator** to convert Out[2] into some recursive relations of $a_{n,i}$.

In[3]:= $\mathbf{rec1 = ApplyOreOperator[Last[ann], a_{n,i}]}$

Out[3]= $\{-(i-n)(m+n+1)a_{n,i} + \left(i(n+2)-(m+4)n-2n^2-2\right)a_{n+1,i} + \left(n^2+3n+2\right)a_{n+2,i}\}$

In[4]:= $\mathbf{rec2 = ApplyOreOperator[First[ann], a_{n,i}]}$

Out[4]= $\{\left(i^2+im-in+n^2+n\right)a_{n,i} - i(i+m+1)(i-n+1)a_{n,i+1} - n(n+1)a_{n+1,i}\}$

We then replace n with $n-1$ for our purpose.

In[5]:= $\mathbf{Simplify\,[Solve\,[(rec1/.\,n \to n-1) == 0, a_{n+1,i}]]}$

Out[5]= $$\left\{\left\{a_{n+1,i} \to \frac{(i-n+1)(m+n)a_{n-1,i} - \left(in+i-mn+m-2n^2\right)a_{n,i}}{n(n+1)}\right\}\right\}$$

In[6]:= $\mathbf{Simplify[Solve[(rec2/.\,n \to n-1) == 0, a_{n-1,i+1}]]}$

Out[6]= $$\left\{\left\{a_{n-1,i+1} \to \frac{\left(i^2+i(m-n+1)+(n-1)n\right)a_{n-1,i} - (n-1)na_{n,i}}{i(i+m+1)(i-n+2)}\right\}\right\}$$

From the outputs, we can obtain recurrence relations (2.1) and (2.2).

We proceed to prove (2.3) and (2.4) by using the command **FindRelation** to find more recurrences. To prove (2.3), we need some recurrences of $a_{n-1,i-1}, a_{n,i}$ and $a_{n-1,i}$ corresponding to operators $1, S[n]S[i]$ and $S[i]$.

In[7]:= $\mathbf{ApplyOreOperator\,[FindRelation[ann, Support \to \{1,\ S[n]S[i],S[i]\}], a_{n,i}]\,;}$

In[8]:= $\mathbf{Simplify[Solve[(\%/.n \to n-1/.i \to i-1) == 0, a_{n-1,i-1}]]}$

Out[8]= $$\left\{\left\{a_{n-1,i-1} \to \frac{(i-n+1)(i+m+n-1)a_{n-1,i} + (n-1)na_{n,i}}{i+m-1}\right\}\right\}$$

It is easy to verify (2.3) for $i = 1$. Similarly, to prove (2.4), the recurrences of $a_{n,i+1}, a_{n,i}$ and $a_{n-1,i}$ corresponding to operators $S[n]S[i], S[n]$ and 1 can be obtained as follows.

In[9]:= $\mathbf{ApplyOreOperator[FindRelation[ann, Support \to \{1,\ S[n]S[i],S[n]\}], a_{n,i}];}$

In[10]:= $\mathbf{Simplify[Solve[(\%/.n \to n-1) == 0, a_{n,i+1}]]}$

Out[10]= $$\left\{\left\{a_{n,i+1} \to \frac{(i+m+n)a_{n,i} - (m+n)a_{n-1,i}}{i(i+m+1)}\right\}\right\}$$

Now we have the required recurrence relations as desired. This completes the proof. ▮

3 Bounds of $\frac{a_{n,i}}{a_{n-1,i}}$

In this section we aim to estimate the upper and lower bounds of $\frac{a_{n,i}}{a_{n-1,i}}$, which compose the main ingredient of our proof of Lemma 3.

To give the bounds of $\frac{a_{n,i}}{a_{n-1,i}}$, we first introduce some notations. Let

$$X(n,i) = 1 - \frac{i+m}{2n} + \frac{m-1}{n+1} + \frac{\sqrt{i^2(n+1)^2 + 2im(n+1)^2 + (m+2n-mn)^2}}{2n(n+1)}, \tag{3.1}$$

$$\begin{aligned} Y(n,i) = 1 &- \frac{2i+m+1}{2(n-1)} + \frac{i+2m}{2n} \\ &+ \frac{\sqrt{i^2(n+1)^2 + 2i((m-1)(n-1)n + 2(m+n)) + (mn-2m-n)^2}}{2(n-1)n}. \end{aligned} \tag{3.2}$$

Lemma 6 *For any positive integers $n \geq 3$, m and $1 \leq i \leq n-2$, we have*

$$X(n,i) \leq \frac{a_{n,i}}{a_{n-1,i}} \leq Y(n,i). \tag{3.3}$$

Proof Fixing $i \geq 1$, we prove this lemma by induction on n for $n \geq i+2$.

For the initial case $n = i+2$, we shall prove

$$X(i+2,i) \leq \frac{a_{i+2,i}}{a_{i+1,i}} = Y(i+2,i).$$

We first give exact formulae for $a_{i+1,i}$ and $a_{i+2,i}$. Our formula for $a_{i+1,i}$ is as follows

$$\begin{aligned} a_{i+1,i} &= \sum_{h=0}^{m-1} \frac{1}{(i-1)!(h+i)(h+i+1)} \binom{h+i+1}{h} \\ &= \frac{1}{(i+1)!} \sum_{h=0}^{m-1} \binom{h+i-1}{h} \\ &= \frac{1}{(i+1)!} \binom{i+m-1}{m-1}, \end{aligned}$$

where the last identity follows from the famous hockey-stick identity. By using Gosper's algorithm, we have

$$\begin{aligned} a_{i+2,i} &= \sum_{h=0}^{m-1} \frac{1}{(i-1)!(h+i)(h+i+1)} \binom{h+i+2}{h} \\ &= \frac{1}{(i+1)(i+2)!} \sum_{h=0}^{m-1} \left(\left(i^2 + i(h+3) + 2\right) \binom{h+i}{h} - \left(i^2 + i(h+2) + 2\right) \binom{h+i-1}{h-1} \right) \\ &= \frac{i^2 + i(m+2) + 2}{(i+1)(i+2)!} \binom{i+m-1}{m-1}. \end{aligned}$$

Therefore, it follows that

$$\frac{a_{i+2,i}}{a_{i+1,i}} = \frac{i^2 + i(m+2) + 2}{(i+1)(i+2)}.$$

By a direct computation, we obtain

$$Y(i+2,i) = \frac{i^2 + i(m+2) + 2}{(i+1)(i+2)} = \frac{a_{i+2,i}}{a_{i+1,i}}.$$

We next use the Mathematica command **Resolve** to prove

$$X(i+2,i) \le \frac{i^2+i(m+2)+2}{(i+1)(i+2)}.$$

The command **Resolve[expr]** can be used to eliminate quantifiers in **expr**. Note that we can also use the command **CylindricalDecomposition** to do this.

In[11]:= $X[n\_,i\_]:=1-\frac{i+m}{2n}+\frac{m-1}{n+1}+\frac{\sqrt{i^2(n+1)^2+2im(n+1)^2+(m+2n-mn)^2}}{2n(n+1)};$

In[12]:= $\text{Resolve}\left[\text{ForAll}\left[\{i,m\}, i\ge 1\,\&\&\,m\ge 1,\ X[i+2,i] \le \frac{i^2+i(m+2)+2}{(i+1)(i+2)}\right]\right]$

Out[12]= True

This proves the inequality for $n=i+2$.

Suppose that the inequality (3.3) holds for the general $n>i+2$, namely,

$$X(n,i) \le \frac{a_{n,i}}{a_{n-1,i}} \le Y(n,i).$$

We proceed to prove the desired inequality holds for $n+1$ as well. By dividing both sides of (2.1) by $a_{n,i}$, we have

$$\frac{a_{n+1,i}}{a_{n,i}} = -\frac{(n-i-1)(m+n)}{n(n+1)}\frac{a_{n-1,i}}{a_{n,i}} - \frac{in+i-mn+m-2n^2}{n(n+1)}.$$

It follows from $-(n-i-1)<0$ that

$$-\frac{(n-i-1)(m+n)}{n(n+1)}\frac{1}{X(n,i)} - \frac{\left(in+i-mn+m-2n^2\right)}{n(n+1)} \le \frac{a_{n+1,i}}{a_{n,i}} \tag{3.4}$$

$$\le -\frac{(n-i-1)(m+n)}{n(n+1)}\frac{1}{Y(n,i)} - \frac{\left(in+i-mn+m-2n^2\right)}{n(n+1)}. \tag{3.5}$$

It is routine to verify that the left-hand side of (3.4) is exactly $X(n,i)$. To complete the induction, we show that $X(n,i)\ge X(n+1,i)$ and the right-hand side of (3.5) is less than or equal to $Y(n+1,i)$ by the Mathematica command **Resolve**.

In[13]:= $\text{Resolve}\left[\text{ForAll}\left[\{n,i,m\}, n\ge i+1\,\&\&\,i\ge 1\,\&\&\,m\ge 1, X[n,i]\ge X[n+1,i]\right]\right]$

Out[13]= True

In[14]:= $Y[n\_,i\_]:=1-\frac{2i+m+1}{2(n-1)}+\frac{i+2m}{2n} + \frac{\sqrt{i^2(n+1)^2+2i((m-1)(n-1)n+2(m+n))+(mn-2m-n)^2}}{2(n-1)n};$

In[15]:= $-\frac{(-i+n-1)(m+n)}{(n(n+1))Y[n,i]}+\frac{-i(n+1)+m(n-1)+2n^2}{n(n+1)} \le Y[n+1,i];$

In[16]:= $\text{Resolve}\left[\text{ForAll}\left[\{n,i,m\}, n\ge i+1\,\&\&\,i\ge 1\,\&\&\,m\ge 1, \%\right]\right]$

Out[16]= True

We finish the induction by **Out**[**13**] and **Out**[**16**]. This completes the proof of Lemma 6. ▮

4 Proof of Lemma 3

To prove Lemma 3, we prove (1.2) by showing that

$$a_{n,i}^2 \geq a_{n,i+1}a_{n,i-1} \tag{4.1}$$

and

$$\frac{a_{n,i+1}}{a_{n-1,i+1}} \geq \frac{a_{n,i}}{a_{n-1,i}} \geq \frac{a_{n+1,i}}{a_{n,i}} \geq \frac{a_{n+1,i-1}}{a_{n,i-1}}. \tag{4.2}$$

We first prove (4.1).

Lemma 7 *For any positive integers $n \geq 4$, m and $1 \leq i \leq n-3$, we have $a_{n,i}^2 \geq a_{n,i+1}a_{n,i-1}$.*

Proof By the recurrence relations (2.2) and (2.3) we have

$$a_{n-1,i}^2 - a_{n-1,i+1}a_{n-1,i-1} = \frac{a_{n-1,i}^2}{i(i+m-1)(i+m+1)(n-i-2)} f_{n,i}\left(\frac{a_{n,i}}{a_{n-1,i}}\right),$$

where

$$\begin{aligned} f_{n,i}(x) =& -n^2(n-1)^2x^2 + n(n-1)^2(m+2n-i-1)x \\ &+ i(i+m-1)(i+m+1)(n-i-2) \\ &- (n-i-1)(i+m+n-1)\left(i^2+i(m-n+1)+(n-1)n\right). \end{aligned}$$

The discriminant of $f_{n,i}(x)$ is

$$\begin{aligned} \Delta_1(n,i) =& (n-1)^2n^2((i^2+(i-1)m+1)^2 + (i+m-1)^2(n-i-2)^2 \\ &+ 2\left(i^2+im+2i+m-1\right)(i+m-1)(n-i-2)). \end{aligned}$$

Since $n \geq i+4$ and $i \geq 1$, it is easy to see that $\Delta_1(n,i) > 0$. Hence, $f_{n,i}(x)$ has two distinct zeros, which are

$$z_1(n,i) = 1 - \frac{i-m+1}{2n} - \frac{\sqrt{\Delta_1(n,i)}}{2(n-1)^2n^2},$$

$$z_2(n,i) = 1 - \frac{i-m+1}{2n} + \frac{\sqrt{\Delta_1(n,i)}}{2(n-1)^2n^2}.$$

Since the leading coefficient of $f_{n,i}(x)$ is negative and $i(i+m-1)(i+m+1)(n-i-2) > 0$, it suffices to show that

$$z_1(n,i) \leq \frac{a_{n,i}}{a_{n-1,i}} \leq z_2(n,i). \tag{4.3}$$

We next prove $\frac{a_{n,i}}{a_{n-1,i}} \geq z_1(n,i)$. In Lemma 6, we have proved that

$$\frac{a_{n,i}}{a_{n-1,i}} \geq X(n,i) = 1 - \frac{i+m}{2n} + \frac{m-1}{n+1} + \frac{\sqrt{i^2(n+1)^2+2im(n+1)^2+(m+2n-mn)^2}}{2n(n+1)}.$$

It is clear that $z_1(n,i) < 1 - \frac{i-m+1}{2n}$. Then $z_1(n,i) \le \frac{a_{n,i}}{a_{n-1,i}}$ can be deduced from

$$\begin{aligned} & 1 - \frac{i-m+1}{2n} - X(n,i) \\ = & \frac{2m+n-1-\sqrt{i^2(n+1)^2+2im(n+1)^2+(m+2n-mn)^2}}{2n(n+1)} \le 0. \end{aligned} \tag{4.4}$$

Indeed, since $i \ge 1$ and $n \ge 4$, we have

$$\begin{aligned} & i^2(n+1)^2 + 2im(n+1)^2 + (m+2n-mn)^2 - (2m+n-1)^2 \\ \ge & (n+1)^2 + 2m(n+1)^2 + (m+2n-mn)^2 - (2m+n-1)^2 \\ = & (n+1)\left(n\left((m-1)^2+3\right) + 6m - 3m^2\right) \\ \ge & 4\left((m-1)^2+3\right) + 6m - 3m^2 \\ = & (m-1)^2 + 15 \ge 0. \end{aligned}$$

Note that the inequality (4.4) can also be proved by cylindrical algebraic decomposition.

We proceed to show $\frac{a_{n,i}}{a_{n-1,i}} \le z_2(n,i)$. Recall that we have proved $\frac{a_{n,i}}{a_{n-1,i}} \le Y(n,i)$ in Lemma 6. It suffices to show that $z_2(n,i) \ge Y(n,i)$, which can be proved by **Mathematica**.

In[17]:= $\mathbf{\Delta_1[n\_\_, i\_\_] := (n-1)^2 n^2 (2(i^2+im+2i+m-1)(i+m-1)(n-i-2) + (i^2+(i-1)m+1)^2 + (i+m-1)^2(n-i-2)^2);}$

In[18]:= $\mathbf{z2[n\_\_, i\_\_] := 1 - \frac{i-m+1}{2n} + \frac{\sqrt{\Delta_1[n,i]}}{2(n-1)^2 n^2};}$

In[19]:= $\mathbf{Resolve[ForAll[\{n,i,m\}, n \ge i+2 \,\&\&\, i \ge 1 \,\&\&\, m \ge 1], z2[n,i] \ge Y[n,i]]}$

Out[19]= True

Hence, we prove $a_{n-1,i}^2 \ge a_{n-1,i+1}a_{n-1,i-1}$ for $1 \le i \le n-4$. Equivalently, we have $a_{n,i}^2 \ge a_{n,i+1}a_{n,i-1}$ for $1 \le i \le n-3$. This completes the proof of Lemma 7. ▮

Next, we prove (4.2) by the following two lemmas.

Lemma 8 *For any positive integers $n \ge 4$, m and $1 \le i \le n-3$, we have*

$$\frac{a_{n,i+1}}{a_{n-1,i+1}} \ge \frac{a_{n,i}}{a_{n-1,i}}, \tag{4.5}$$

$$\frac{a_{n+1,i}}{a_{n,i}} \ge \frac{a_{n+1,i-1}}{a_{n,i-1}}. \tag{4.6}$$

Proof We first prove (4.5), which is equivalent to

$$a_{n,i+1}a_{n-1,i} - a_{n,i}a_{n-1,i+1} \ge 0.$$

Inequality (4.6) can then be deduced from (4.5).

Utilizing the recurrence relations (2.2) and (2.4), we have

$$\begin{aligned} & i(i+m+1)(n-i-2)(a_{n,i+1}a_{n-1,i}-a_{n,i}a_{n-1,i+1}) \\ = & -(n-1)na_{n,i}^2-\left(in+i-mn+2m-2n^2+3n\right)a_{n,i}a_{n-1,i}-(n-i-2)(m+n)a_{n-1,i}^2. \end{aligned}$$

Let

$$g_{n,i}(x)=-(n-1)nx^2-\left(in+i-mn+2m-2n^2+3n\right)x-(n-i-2)(m+n).$$

Then

$$i(i+m+1)(n-i-2)(a_{n,i+1}a_{n-1,i}-a_{n,i}a_{n-1,i+1})=a_{n-1,i}^2 g_{n,i}\left(\frac{a_{n,i}}{a_{n-1,i}}\right).$$

Since $i(i+m+1)(n-i-2)>0$ and $a_{n-1,i}^2>0$, it remains to show that $g_{n,i}(\frac{a_{n,i}}{a_{n-1,i}})\geq 0$. The discriminant of $g_{n,i}(x)$ is

$$\Delta_2(n,i)=i^2(n+1)^2+2i((m-1)(n-1)n+2(m+n))+(mn-2m-n)^2,$$

which is manifestly positive. Then $g_{n,i}(x)$ has two distinct zeros $y_1(n,i)$ and $y_2(n,i)$, where

$$y_1(n,i)=1-\frac{2i+m+1}{2(n-1)}+\frac{i+2m}{2n}-\frac{\sqrt{\Delta_2(n,i)}}{2(n-1)n},$$

$$y_2(n,i)=1-\frac{2i+m+1}{2(n-1)}+\frac{i+2m}{2n}+\frac{\sqrt{\Delta_2(n,i)}}{2(n-1)n}.$$

Since the leading coefficient of $g_{n,i}(x)$ is negative, the statement that $g_{n,i}(\frac{a_{n,i}}{a_{n-1,i}})\geq 0$ is equivalent to

$$y_1(n,i)\leq\frac{a_{n,i}}{a_{n-1,i}}\leq y_2(n,i)$$

by the property of the quadratic function.

Note that $y_2(n,i)=Y(n,i)$, and we have proved that $\frac{a_{n,i}}{a_{n-1,i}}\leq Y(n,i)$ in Lemma 6. Therefore, we have $\frac{a_{n,i}}{a_{n-1,i}}\leq y_2(n,i)$. It remains to show that $y_1(n,i)\leq\frac{a_{n,i}}{a_{n-1,i}}$.

Recall that

$$a_{n,i}=\sum_{h=0}^{m-1}\frac{\binom{h+n}{h}}{(i-1)!(h+i)(h+i+1)}.$$

Since each term in the sum of $a_{n,i}$ is positive and

$$\binom{h+n}{h}\geq\binom{h+n-1}{h},$$

we have

$$\frac{a_{n,i}}{a_{n-1,i}}\geq 1.$$

We shall show that $y_1(n,i)\leq 1$. Since $i\geq 1$, it is easy to check that

$$y_1(n,i)-1\leq y_1(n,1)-1=\frac{m(n-2)-2n-1-\sqrt{m^2(n-2)^2+2m(n+2)+8n+1}}{2(n-1)n}.$$

and

$$m^2(n-2)^2 + 2m(n+2) + 8n + 1 - (m(n-2) - 2n - 1)^2 = 4(m-1)(n-1)n \geq 0.$$

Then $y_1(n,i) \leq 1$. Hence, we have $y_1(n,i) \leq \frac{a_{n,i}}{a_{n-1,i}}$ which completes the proof of (4.5).

Replacing n by $n+1$ and i by $i-1$ in (4.5), we have

$$\frac{a_{n+1,i}}{a_{n,i}} \geq \frac{a_{n+1,i-1}}{a_{n,i-1}}$$

for $2 \leq i \leq n-1$. It remains to show that (4.6) for $i = 1$. Recall that $\frac{a_{n,i}}{a_{n-1,i}} \geq 1$. The desired inequality follows from $a_{n,0} = a_{n-1,0} = 1$. This completes the proof. ∎

Lemma 9 *For any positive integers $n \geq 4$, m and $1 \leq i \leq n-3$, we have*

$$\frac{a_{n,i}}{a_{n-1,i}} \geq \frac{a_{n+1,i}}{a_{n,i}}. \tag{4.7}$$

Proof In the proof of Lemma 6, we have shown that

$$\frac{a_{n+1,i}}{a_{n,i}} = -\frac{(n-i-1)(m+n)}{n(n+1)}\frac{a_{n-1,i}}{a_{n,i}} - \frac{in+i-mn+m-2n^2}{n(n+1)}.$$

Thus, (4.7) turns out to be

$$\frac{a_{n,i}}{a_{n-1,i}} \geq -\frac{(n-i-1)(m+n)}{n(n+1)}\frac{a_{n-1,i}}{a_{n,i}} - \frac{in+i-mn+m-2n^2}{n(n+1)}.$$

Multiplying both sides of the above inequality by $\frac{a_{n,i}}{a_{n-1,i}}$, we get that

$$\left(\frac{a_{n,i}}{a_{n-1,i}}\right)^2 \geq \frac{(i-n+1)(m+n)}{n(n+1)} - \frac{in+i-mn+m-2n^2}{n(n+1)}\frac{a_{n,i}}{a_{n-1,i}}.$$

Let

$$x_1(n,i) = 1 - \frac{i+m}{2n} + \frac{m-1}{n+1} - \frac{\sqrt{i^2(n+1)^2 + 2im(n+1)^2 + (m+2n-mn)^2}}{2n(n+1)}$$

and

$$x_2(n,i) = 1 - \frac{i+m}{2n} + \frac{m-1}{n+1} + \frac{\sqrt{i^2(n+1)^2 + 2im(n+1)^2 + (m+2n-mn)^2}}{2n(n+1)}$$

be the two real roots of the equation

$$x^2 = \frac{(i-n+1)(m+n)}{n(n+1)} - \frac{in+i-mn+m-2n^2}{n(n+1)}x.$$

Therefore, it suffices to show that $\frac{a_{n,i}}{a_{n-1,i}} \geq x_2(n,i)$ by the property of the quadratic function. Note that $x_2(n,i)$ is just $X(n,i)$ defined in (3.1) and we have proved $\frac{a_{n,i}}{a_{n-1,i}} \geq X(n,i)$ by Lemma 6. Thus we finish the proof of (4.7). ∎

Now, we are able to prove Lemma 3.

Proof of Lemma 3 We prove this lemma by showing that inequality (1.2) holds. For $1 \leq i \leq n-3$, it follows from Lemma 8 that

$$\frac{a_{n,i+1}}{a_{n-1,i+1}} \geq \frac{a_{n,i}}{a_{n-1,i}} \quad \text{and} \quad \frac{a_{n+1,i}}{a_{n,i}} \geq \frac{a_{n+1,i-1}}{a_{n,i-1}}.$$

By Lemma 9, we have

$$\frac{a_{n,i}}{a_{n-1,i}} \geq \frac{a_{n+1,i}}{a_{n,i}}.$$

Thus, we have

$$\frac{a_{n,i+1}}{a_{n-1,i+1}} \geq \frac{a_{n,i}}{a_{n-1,i}} \geq \frac{a_{n+1,i}}{a_{n,i}} \geq \frac{a_{n+1,i-1}}{a_{n,i-1}}.$$

Then

$$\frac{a_{n,i+1}}{a_{n-1,i+1}} \geq \frac{a_{n+1,i-1}}{a_{n,i-1}},$$

namely,

$$a_{n,i+1}a_{n,i-1} \geq a_{n-1,i+1}a_{n+1,i-1}.$$

By Lemma 7, we have

$$a_{n,i}^2 \geq a_{n,i+1}a_{n,i-1}.$$

Therefore, we have

$$a_{n,i}^2 \geq a_{n-1,i+1}a_{n+1,i-1}.$$

This completes the proof of Lemma 3. ▮

5 Proof of Theorem 2

In this section, we first give a proof of Lemma 4, and then prove Theorem 2.

Proof of Lemma 4 By direct computation, we have

$$\begin{aligned}
\frac{b_{d,i}^2}{b_{d,i+1}b_{d,i-1}} &= \frac{(i+1)(d-2i)(d-2i+1)(d-i-1)(d-i+m+1)}{i(d-2i-1)(d-i)(d-2(i+1))(d-i+m)} \\
&= \frac{(i+1)(d-2i)(d-i+m+1)}{i(d-2i-1)(d-i+m)} \frac{(d-2i+1)(d-i-1)}{(d-i)(d-2i-2)} \\
&\geq \frac{(d-2i+1)(d-i-1)}{(d-i)(d-2i-2)} \\
&= 1 + \frac{2d-i-1}{(d-i)(d-2(i+1))} \geq 1.
\end{aligned}$$

This completes the proof of the log-concavity of $b_{n,i}$. ▮

We proceed to prove Theorem 2.

Proof of Theorem 2 By Lemma 3, we have $a_{n,i}^2 \geq a_{n-1,i+1}a_{n+1,i-1}$. Let $n = d-i$, we have

$$a_{d-i,i}^2 \geq a_{d-i-1,i+1}a_{d-i+1,i-1}.$$

Recall that we have shown $b_{d,i}^2 \geq b_{d,i+1}b_{d,i-1}$ in Lemma 4. Since $c_{m,d,i} = b_{d,i}a_{d-i,i}$, we have $c_{m,d,i}^2 \geq c_{m,d,i+1}c_{m,d,i-1}$. This completes the proof.

▮

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