The log-concavity of Kazhdan-Lusztig polynomials of thagomizer matroids

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Abstract. Elias, Proudfoot, and Wakefield conjectured that the Kazhdan-Lusztig polynomial of any matroid is log-concave. In this paper, we prove that the Kazhdan-Lusztig polynomials of thagomizer matroids are log-concave.

Keywords: Log-concavity; Kazhdan-Lusztig polynomials; thagomizer matroids; recurrence relation.

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1 Introduction

The objective of this paper is to prove the log-concavity of Kazhdan-Lusztig polynomials of thagomizer matroids.

Let $\chi_M(t)$ be the characteristic polynomial of a matroid M. Elias, Proudfoot, and Wakefield [2] associated to every matroid M a polynomial $P_M(t)$ with integer coefficients such that the following conditions are satisfied:

1. If $\operatorname{rk} M = 0$, then $P_M(t) = 1$.

2. If $\operatorname{rk} M > 0$, then $\deg P_M(t) < \frac{1}{2}\mathrm{rk}M$, where $\mathrm{rk}M$ is the rank of M.

3. For every M, $t^{\operatorname{rk} M} P_M\left(t^{-1}\right) = \sum_F \chi_{M_F}(t) P_{M^F}(t)$, where M_F and M^F denotes the restriction and contraction on F respectively.

The polynomial $P_M(t)$ are usually called the Kazhdan-Lusztig polynomial of M. They proposed a conjecture which states that the coefficients of these polynomials are always non-negative. This conjecture was recently confirmed by Braden, Huh, Matherne, Proudfoot, and Wang [1].

Elias, Proudfoot, and Wakefield also conjectured the log-concavity of these polynomials. This conjecture is still open. We say that a finite sequence $\{a_i\}_{i=0}^n$ is *log-concave* if the consecutive numbers satisfy $a_i^2 \geq a_{i-1}a_{i+1}$ for $1 \leq i \leq n-1$, and a polynomial $\sum_{i=0}^n a_i x^i$ is *log-concave* if its coefficients form a log-concave sequence.

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Conjecture 1.1 ([4]). *For any matroid M, the Kazhdan-Lusztig polynomial $P_M(t)$ is log-concave.*

Conjecture 1.1 has been confirmed for uniform matroids, whirl matroids, wheel matroids, and graphic matroids of cycle graghs, fan graghs, and squares of paths [4, 6, 7, 11].

In this paper, we will prove this conjecture for thagomizer matroids. Consider a complete bipartite graph $K_{2,n}$ and let T_n be the graph obtained by joining the two distinguished vertices with an edge. The graph T_n is called a thagomizer graph. Let $P_{T_n}(t)$ be the Kazhdan-Lusztig polynomials of its graphical matroid.

Theorem 1.2. *For any positive integer n, the Kazhdan-Lusztig polynomial $P_{T_n}(t)$ is log-concave.*

We prove Theorem 1.2 through the following ideas. Let $c_{n,k}$ be the coefficient of t^k in $P_{T_n}(t)$. Gedeon [3] gave a very nice formula for $c_{n,k}$ as follows:

$$c_{n,k} = \frac{1}{n+1}\binom{n+1}{k}\sum_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n-k+1}{n-j}. \tag{1}$$

An equivalent generalization of this formula was conjectured by Gedeon [3] and proved by Xie and Zhang[12]. Let

$$d_{n,k} = \sum_{j=2k}^{n}\binom{j-k-1}{k-1}\binom{n-k+1}{n-j}$$

and

$$b_{n,k} = \frac{1}{n+1}\binom{n+1}{k}.$$

Consequently,

$$c_{n,k} = b_{n,k}d_{n,k}.$$

We shall prove the log-concavity of $c_{n,k}$ by showing that the same property holds for $d_{n,k}$ and $b_{n,k}$ respectively. The log-concavity of $b_{n,k}$ is easy to check. The key ingredient of our proof is the following lemma, which confirms the log-concavity of $d_{n,k}$.

Lemma 1.3. *For any positive integers n and k with $n \geq 2k+1$, we have*

$$d_{n,k}^2 \geq d_{n,k+1}d_{n,k-1}.$$

Our proof of Lemma 1.3 is inspired by Kauers and Paule's computer proof of Moll's log-concavity conjecture [5].

The outline of the rest of the paper is as follows. We first give some recurrence relations of $d_{n,k}$ in Section 2. We next estimate upper and lower bounds of $\frac{d_{n,k}}{d_{n-1,k}}$ in Section 3. Finally, we prove Lemma 1.3 and Theorem 1.2 in Section 4.

2 Recurrence relations of $d_{n,k}$

In this section we give some recurrence relations of $d_{n,k}$, which will be used later. To do this, we use Zeilberger's algorithm[10, 13], which is an effective tool for proving combinatorial identities especially including sums of hypergeometric terms. We shall briefly introduce how Zeilberger's algorithm works to deduce some recurrence relations. Let $F(n,k)$ be a double hypergeometric term, namely, both $F(n+1,k)/F(n,k)$ and $F(n,k+1)/F(n,k)$ are rational functions of n and k. Zeilberger's algorithm is devised to find a double hypergeometric term $G(n,k)$ and polynomials $a_0(n), a_1(n), \cdots, a_m(n)$ which are independent of k such that

$$a_0(n)F(n,k) + \cdots + a_m(n)F(n+m,k) = G(n,k+1) - G(n,k). \tag{2}$$

Set $S(n) = \sum_{k=a}^{b} F(n,k)$. Summing (2) over k ranging from a to b, where $a \le b$. It follows from (2) that

$$a_0(n)S(n) + \cdots + a_m(n)S(n+m) = G(n,b+1) - G(n,a).$$

Note that $G(n,b+1) - G(n,a) = 0$ holds for many sums arising in applications.

Lemma 2.1. *For any positive integers n and k with $n \ge 2k+1$, we have*

$$d_{n+1,k} = \frac{(-2k+2n+2)d_{n-1,k} + (4k-3n-2)d_{n,k}}{2k-n-1}, \tag{3}$$

$$d_{n-1,k-1} = \frac{(-2k+2n+2)d_{n-1,k} - nd_{n,k}}{2k-n-1}, \tag{4}$$

$$d_{n-1,k+1} = \frac{\left(-4k^2+2k(n-1)+n(n+1)\right)d_{n-1,k} + n(2k-n)d_{n,k}}{4k(k-n)}. \tag{5}$$

Proof. We first prove (3) by Zeilberger's algorithm. Let

$$F_1(j,n) = \binom{j-k-1}{k-1}\binom{n-k+1}{n-j}$$

and

$$G_1(j,n) = -\frac{(n-k+2)!}{(j-k)(j-2k-1)!(k-1)!(n-j+2)!}.$$

Clearly, $d_{n,k} = \sum_{j=2k}^{n} F_1(j,n)$. It follows from the convention $\frac{1}{(-1)!} = 0$ that $G_1(n+3,n) = G_1(2k,n) = 0$. By using Zeilberger's algorithm on n, we obtain

$$\begin{aligned} -2(k-n-2)F_1(j,n) + (4k-3n-5)F_1(j,n+1) \\ + (-2k+n+2)F_1(j,n+2) = G_1(j+1,n) - G_1(j,n). \end{aligned}$$

By summing the above equation over j from $2k$ to $n+2$, we get

$$-2(k-n-2)d_{n,k}+(4k-3n-5)d_{n+1,k}+(-2k+n+2)d_{n+2,k}=0. \tag{6}$$

Thus we can get (3) by replacing n with $n-1$ in (6).

We next prove

$$d_{n-1,k+1}=-\frac{\left(8k^2-8kn-2k+n^2+n\right)d_{n-1,k}+\left(4k^2-4kn-2k+n^2+n\right)d_{n-1,k-1}}{4k(k-n)} \tag{7}$$

by following similar lines to (3). Let

$$F_2(j,k)=\binom{j-k-1}{k-1}\binom{n-k+1}{n-j}$$

and

$$G_2(j,k)=\frac{B(j,k)(n-k)!}{(j-k-2)(j-k-1)(k-j)(j-2k-1)!(n-j)!k!},$$

where

$$\begin{aligned}B(j,k)=&\,2j^4+j^3(n-16k-13)+j^2(44k^2-3kn+71k-3n+25)\\&-j(k^2n+48k^3+117k^2-kn^2-5kn+78k-2n+14)\\&-k^2n^2+3k^3n+6k^2n+18k^4+51k^3+49k^2-2kn^2+14k.\end{aligned}$$

Clearly, $d_{n,k}=\sum_{j=2k}^{n}F_2(j,k)$. We get $G_2(n+1,k)=G_2(2k,k)=0$. By using Zeilberger's algorithm on k, we obtain

$$\begin{aligned}-(2k-n)(2k-n+1)F_2(j,k)+(-8k^2+8kn-6k-n^2+5n)F_2(j,k+1)&\\-4(k+1)(k-n)F_2(j,k+2)=G_2(j+1,k)-G_2(j,k).&\end{aligned}$$

By summing the above equation over j from $2k$ to n, we get

$$\begin{aligned}-(2k-n)(2k-n+1)d_{n,k}+(-8k^2+8kn-6k-n^2+5n)d_{n,k+1}&\\-4(k+1)(k-n)d_{n,k+2}=0.&\end{aligned} \tag{8}$$

Thus we can get (7) by replacing k with $k-1$ and n with $n-1$ in (8).

We proceed to prove

$$d_{n+1,k}=2d_{n,k}+d_{n-1,k-1}. \tag{9}$$

Replacing the summation index j in $d_{n,k}$ with $i=n-j$, we have

$$d_{n,k}=\sum_{i=0}^{n-2k}\binom{n-i-k-1}{k-1}\binom{n-k+1}{i}.$$

Thus we have

$$\begin{aligned}
d_{n+1,k} - d_{n-1,k-1} - d_{n,k} &= \sum_{i=0}^{n-2k+1} \binom{n-i-k}{k-1}\binom{n-k+2}{i} \\
&\quad - \left(\sum_{i=0}^{n-2k+1} \binom{n-i-k-1}{k-2}\binom{n-k+1}{i} \right. \\
&\qquad \left. + \sum_{i=0}^{n-2k} \binom{n-i-k-1}{k-1}\binom{n-k+1}{i} \right) \\
&= \sum_{i=0}^{n-2k+1} \binom{n-i-k}{k-1}\binom{n-k+2}{i} - \sum_{i=0}^{n-2k+1} \binom{n-i-k}{k-1}\binom{n-k+1}{i} \\
&= \sum_{i=1}^{n-2k+1} \binom{n-i-k}{k-1}\binom{n-k+1}{i-1} \\
&= \sum_{i=0}^{n-2k} \binom{n-i-k-1}{k-1}\binom{n-k+1}{i} = d_{n,k}.
\end{aligned}$$

This completes the proof of (9).

Therefore, we subtract (9) from (3) to obtain the recurrence relation (4). Finally, by substituting (4) for $d_{n-1,k-1}$ in (7), we obtain the recurrence relation (5). Now all the required recurrence relations have been obtained. This completes the proof. □

Note that the Mathematica program of finding the recurrence relations (3) and (6) are described in Appendix. We use the package **fastZeil** developed by Paule, Schorn, and Riese[8, 9] which is a Mathematica implementation of the Zeilberger's algorithm.

3 Bounds of $\frac{d_{n,k}}{d_{n-1,k}}$

In this section we estimate the upper and lower bounds of $\frac{d_{n,k}}{d_{n-1,k}}$. To give the bounds of $\frac{d_{n,k}}{d_{n-1,k}}$, we first introduce some notations. Let

$$X(n) = \frac{1}{n} + 1, \tag{10}$$

$$Y(n,k) = \frac{2k}{n-2k} + \frac{2}{n} + 2. \tag{11}$$

Lemma 3.1. *For any positive integers n and k with $n \geq 2k+1$, we have*

$$X(n) \leq \frac{d_{n,k}}{d_{n-1,k}} \leq Y(n,k).$$

Proof. Fixing k, we prove this lemma by using induction on n. For the case $n = 2k+1$, we need to show

$$X(2k+1) \le \frac{d_{2k+1,k}}{d_{2k,k}} \le Y(2k+1,k).$$

By the definition,

$$\frac{d_{2k+1,k}}{d_{2k,k}} = 2k+2,$$

$$X(2k+1) = \frac{1}{2k+1} + 1,\ Y(2k+1,k) = 2k + \frac{2}{2k+1} + 2.$$

Therefore, the desired inequality holds for $n = 2k+1$.

Assume that the inequality holds for some $n \ge 2k+1$, namely,

$$X(n) \le \frac{d_{n,k}}{d_{n-1,k}} \le Y(n,k).$$

We proceed to prove the desired inequalities hold for $n+1$ as well. By dividing both sides of (3) by $d_{n,k}$, we have

$$\frac{d_{n+1,k}}{d_{n,k}} = \frac{(-2k+2n+2)d_{n-1,k}}{(2k-n-1)d_{n,k}} + \frac{4k-3n-2}{2k-n-1}.$$

Since $2k-n-1 \le -2 < 0$ we obtain

$$\frac{-2k+2n+2}{(2k-n-1)X(n)} + \frac{4k-3n-2}{2k-n-1} \le \frac{d_{n+1,k}}{d_{n,k}} \tag{12}$$

$$\le \frac{-2k+2n+2}{(2k-n-1)Y(n,k)} + \frac{4k-3n-2}{2k-n-1}. \tag{13}$$

By the definition of $X(n)$, we can see the left side of (12) is equal to $X(n+1)$. Indeed,

$$\begin{aligned}\frac{-2k+2n+2}{(2k-n-1)X(n)} + \frac{4k-3n-2}{2k-n-1} &= \frac{(-2k+2n+2)n}{(2k-n-1)(n+1)} + \frac{4k-3n-2}{2k-n-1}\\ &= \frac{1}{n+1} + 1 = X(n+1).\end{aligned}$$

We only need to show the right side of (13) is less than or equal to $Y(n+1,k)$, namely,

$$\frac{-2k+2n+2}{(2k-n-1)Y(n,k)} + \frac{4k-3n-2}{2k-n-1} \le Y(n+1,k).$$

By the definition of $Y(n,k)$, it suffices to verify

$$\frac{-2k+2n+2}{(2k-n-1)\left(\frac{2k}{n-2k} + \frac{2}{n} + 2\right)} + \frac{4k-3n-2}{2k-n-1} \le \frac{2k}{n+1-2k} + \frac{2}{n+1} + 2.$$

which can be reduced to

$$\frac{2(2k^2 - 2k(n+1)^2 + n(n+1)^2)}{(n+1)(n+1-2k)(kn+2k-n^2-n)} \leq 0.$$

Since $n \geq 2k+1$, we get $n+1-2k \geq 2 > 0$ and

$$kn + 2k - n^2 - n \leq \frac{n-1}{2}(n+2) - n(n+1) = -\frac{1}{2}n(n+1) - 1 < 0.$$

We proceed to prove

$$2k^2 - 2k(n+1)^2 + n(n+1)^2 > 0.$$

Let $f(k) = 2k^2 - 2k(n+1)^2 + n(n+1)^2$. By considering

$$f(k) = 2\left(k - \frac{(n+1)^2}{2}\right)^2 - \frac{1}{2}(n+1)^2\left(n^2+1\right)$$

as a quadratic function of k, the symmetry axis of $f(k)$ is $k = \frac{(n+1)^2}{2}$. Since $n \geq 2k+1$, we have $k \leq \frac{n-1}{2} < \frac{n+1}{2} < \frac{(n+1)^2}{2}$, and therefore

$$\begin{aligned} f(k) \geq f(\frac{n-1}{2}) &= \frac{1}{2}(n^2+n+2)^2 - \frac{1}{2}(n+1)^2(n^2+1) \\ &= \frac{1}{2}(3n^2+2n+3) > 0. \end{aligned}$$

This completes the proof. □

4 Proof of Theorem 1.2

We first prove Lemma 1.3.

Proof of Lemma 1.3. For the convenience of notation, we shall prove that

$$d_{n-1,k}^2 - d_{n-1,k+1}d_{n-1,k-1} \geq 0.$$

Through recurrence relations (4) and (5), we get that

$$\begin{aligned} &4k(k-n)(2k-n-1)\left(d_{n-1,k}^2 - d_{n-1,k+1}d_{n-1,k-1}\right) \\ =&\left((n+1)d_{n-1,k} - nd_{n,k}\right)\left(2(k(n+2) - n(n+1))d_{n-1,k} + n(n-2k)d_{n,k}\right). \end{aligned}$$

Let

$$f_{n,k}(x) = \left((n+1) - nx\right)\left(2(k(n+2) - n(n+1) + n(n-2k)x)\right).$$

Hence it follows that

$$4k(k-n)(2k-n-1)\left(d_{n-1,k}^2 - d_{n-1,k+1}d_{n-1,k-1}\right) = d_{n-1,k}^2 f_{n,k}\left(\frac{d_{n,k}}{d_{n-1,k}}\right).$$

It is easy to see the two distinct zeros of $f_{n,k}(x)$ are just $X(n)$ and $Y(n,k)$ defined in (10) and (11) respectively. Since the leading coefficient of $f_{n,k}(x)$ is negative, the log-concavity follows from Lemma 3.1. This completes the proof. $\square$

We proceed to prove Theorem 1.2.

Proof of Theorem 1.2. According to Lemma 1.3, we have $d_{n,k}^2 \geq d_{n,k+1}d_{n,k-1}$. It is well known that the binomial coefficients satisfy

$$\binom{n}{k}^2 \geq \binom{n}{k+1}\binom{n}{k-1}.$$

Thus it follows that $b_{n,k}^2 \geq b_{n,k+1}b_{n,k-1}$. Therefore, since $c_{n,k} = b_{n,k}d_{n,k}$, we have that $c_{n,k}^2 \geq c_{n,k+1}c_{n,k-1}$. $\square$

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Appendix

Here we show the Mathematica program how to obtain the recurrence relations (3) and (7). The Zeilberger's algorithm was implemented as the function

Zb[function, range, n, order]

in the package **fastZeil** to find a recurrence relation of given order in n for the sum of the function over the range.

In[1]:= **<< RISC\`fastZeil\`**

Fast Zeilberger Package version 3.61
written by Peter Paule, Markus Schorn, and Axel Riese
Copyright Research Institute for Symbolic Computation (RISC),
Johannes Kepler University, Linz, Austria

In[2]:= $\mathbf{Zb[Binomial[}\boldsymbol{j-k-1,k-1}\mathbf{]Binomial[}\boldsymbol{n-k+1,n-j}\mathbf{],\{}\boldsymbol{j,2k,n}\mathbf{\},}\boldsymbol{n}\mathbf{,2]}$

If '-2k + n' is a natural number and none of -1 + k, 1 - k + n is a negative integer, then:

Out[2]= $\{-2(-2+k-n)\mathrm{SUM}[n]+(-5+4k-3n)\mathrm{SUM}[1+n]+(2-2k+n)\mathrm{SUM}[2+n]==0\}$

In[3]:= $\mathbf{Zb[Binomial[}\boldsymbol{j-k-1,k-1}\mathbf{]Binomial[}\boldsymbol{n-k+1,n-j}\mathbf{],\{}\boldsymbol{j,2k,n}\mathbf{\},}\boldsymbol{k}\mathbf{,2]}$

If '-2 - 2k + n' is a natural number and none of -1 + k, -4 - 2k + n, -1 - k + n is a negative integer, then:

Out[3]= $\{-(2k-n)(1+2k-n)\mathrm{SUM}[k]+(-6k-8k^2+5n+8kn-n^2)\mathrm{SUM}[1+k]-4(1+k)(k-n)\mathrm{SUM}[2+k]==0\}$