STATISTICS ON MULTIPERMUTATIONS AND PARTIAL γ-POSITIVITY

ZHICONG LIN, JUN MA, AND PHILIP B. ZHANG

Abstract. We prove that the enumerative polynomials of Stirling multipermutations by the statistics of plateaux, descents and ascents are partial γ-positive. Specialization of our result to the Jacobi-Stirling permutations confirms a recent partial γ-positivity conjecture due to Ma, Yeh and the second named author. Our partial γ-positivity expansion, as well as a combinatorial interpretation for the corresponding γ-coefficients, are obtained via the machine of context-free grammars and a group action on Stirling multipermutations. Besides, we also provide an alternative approach to the partial γ-positivity from the stability of certain multivariate polynomials on Stirling multipermutations. Moreover, we prove the partial γ-positivity for the enumerators of multipermutations by plateaux, descents and ascents via introducing a group action on words. Since multipermutations without any plateau are Smirnov words, our result generalizes a γ-positivity result due to Linusson, Shareshian and Wachs in this special case. Interestingly, our second action on multipermutations applies also to Stirling multipermutations and results in another combinatorial expansion for their partial γ-positivity. Finally, using a modification of our second group action and Foata's first fundamental transformation, we prove the partial γ-positivity for the enumerators of multipermutations by fixed points, excedances and drops, generalizing another result of Linusson, Shareshian and Wachs for derangements of a multiset.

1. Introduction

Let $\mathcal{A}$ be an alphabet whose elements are totally ordered. For a word $w = w_1 \cdots w_n \in \mathcal{A}^n$, an index i, $0 \le i \le n$, is an *ascent* (resp. a *plateau*, a *descent*) of w if $w_i < w_{i+1}$ (resp. $w_i = w_{i+1}$, $w_i > w_{i+1}$), where we use the convention that $w_0 = w_{n+1} = -\infty$. Here $-\infty$ is considered as an extra element smaller than all letters in $\mathcal{A}$. For instance, if $w = 11211 \in \mathbb{P}^5$, then $0, 2$ are ascents, $1, 4$ are plateaux and $3, 5$ are descents of w. Let $\mathsf{asc}(w)$ (resp. $\mathsf{plat}(w)$, $\mathsf{des}(w)$) be the number of ascents (resp. plateaux, descents) of w. This work is motivated by

- a partial γ-positivity conjecture of Ma, Ma and Yeh [35] concerning the study of these three statistics on the so-called *Jacobi–Stirling permutations* introduced in [24];
- and two γ-positivity expansion formulae of Linusson, Shareshian and Wachs [33] for the descent polynomials on Smirnov words and for the excedance polynomials on multiset derangements, which were established using Rees products of posets.

The purpose of this paper is to provide several proofs of the partial γ-positivity conjecture for Jacobi–Stirling permutations and generalize Linusson–Shareshian–Wachs' result to the

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context of partial γ-positivity via introducing a word version of the *Foata–Strehl group actions* [21] on permutations.

Gamma-positive polynomials arise frequently in enumerative combinatorics and have recent impetus coming from enumerative geometry [23, 38] and poset homology [33]; see also the survey of Athanasiadis [4]. A univariate polynomial $f(x)$ is said to be *γ-positive* if it can be expanded as

$$f(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\gamma_k x^k(1+x)^{n-2k}$$

with $\gamma_k\geq 0$. A bivariate polynomial $h(x,y)$ is said to be *homogeneous γ-positive*, if $h(x,y)$ is homogeneous and $h(x,1)$ is *γ-positive*. This is equivalent to say that $h(x,y)$ can be expressed as

$$h(x,y)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\gamma_k (xy)^k(x+y)^{n-2k}$$

with $\gamma_k\geq 0$. Alternatively, $h(x,y)$ is a symmetric function in x and y which is *e-positive*, i.e., $h(x,y)$ can be written as a non-negative linear combination of the elementary symmetric functions $e_{2^k1^{2n-k}}(x,y)$. One of the typical examples arising from permutation statistics due to Foata and Schüzenberger [20] is the *bivariate Eulerian polynomial* $A_n(x,y)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathsf{asc}(\pi)}y^{\mathsf{des}(\pi)}$, where $\mathfrak{S}_n$ is the set of all permutations of $[n]:=\{1,2,\ldots,n\}$.

A trivariate polynomial $p(x,y,z)=\sum_i s_i(x,y)z^i$ is said to be *partial γ-positive* if every $s_i(x,y)$ is homogeneous γ-positive. The first example of partial γ-positive polynomial that we can find in the literature (see [32, 40]) is the *trivariate Eulerian polynomial*

$$A_n(x,y,z)=\sum_{\pi\in\mathfrak{S}_n}x^{\mathsf{exc}(\pi)}y^{\mathsf{drop}(\pi)}z^{\mathsf{fix}(\pi)},$$

where $\mathsf{exc}(\pi)$ (resp. $\mathsf{drop}(\pi)$, $\mathsf{fix}(\pi)$) denotes the number of *excedances* (resp. *drops*, *fixed points*) of π. Note that $A_n(x,y)=A_n(x,y,y)$ by a fundamental bijection on $\mathfrak{S}_n$ (see [42, Sec. 1.3]). Recently, by the change of grammars, Ma, Ma and Yeh [35] investigated the partial γ-positivity of the distribution polynomials of ascents, descents and plateaux over Stirling permutations introduced by Gessel and Stanley [25]. Here we will extend their result to Stirling permutations of a multiset.

Let us first give an overview of the Stirling permutations. It is well known [20] that the n-th Eulerian polynomials $A_n(t):=A_n(1,t)$ satisfies Euler's classical identity

$$\sum_{k\geq 1}k^nt^k=\frac{A_n(t)}{(1-t)^{n+1}}.$$

Recall that the *Stirling number of the second kind* $S(n,k)$ enumerates the set partitions of $[n]$ with k blocks. In order to interpret the *second-order Eulerian polynomials* $C_n(t)$ appearing as

$$\sum_{k\geq 0}S(n+k,k)t^k=\frac{C_n(t)}{(1-t)^{2n+1}},\tag{1.1}$$

Gessel and Stanley [25] invented the Stirling permutations. A *Stirling permutation* of order n is a permutation of the multiset $\{1,1,2,2,\ldots,n,n\}$ such that for each $i\in[n]$, all entries between the two occurrences of i are larger than i. Gessel and Stanley [25] provided three different proofs for the interpretation

$$C_n(t)=\sum_{\pi\in\mathcal{Q}_n} t^{\mathsf{des}(\pi)},$$

where $\mathcal{Q}_n$ denotes the set of Stirling permutations of order n. Interestingly, the three statistics asc, plat and des are equidistributed over $\mathcal{Q}_n$, as was shown by Bóna in [7] where the statistic plat was first considered.

The Stirling permutations extends naturally to permutations of a general multiset. For each vector $\mathbf{m}=(m_1,m_2,\ldots,m_n)\in\mathbb{P}^n$, denote by $\mathfrak{S}_{\mathbf{m}}$ the set of all permutations of the general multiset $\{1^{m_1},2^{m_2},\ldots,n^{m_n}\}$, where i appears m_i times. A multipermutation in $\mathfrak{S}_{\mathbf{m}}$ is a *generalized Stirling permutation* (or *Stirling multipermutation*) if all entries between any two occurrences of i are larger than i for each $i\in[n]$. Let $\mathcal{Q}_{\mathbf{m}}$ be the set of all generalized Stirling permutations in $\mathfrak{S}_{\mathbf{m}}$. Note that $\mathcal{Q}_{\mathbf{m}}=\mathcal{Q}_n$ when $\mathbf{m}=(2,2,\ldots,2)$. The generalized Stirling permutations and various statistics over them have been studied in [12, 26, 28]. In particular, Brenti [12] showed that the descent polynomial over $\mathcal{Q}_{\mathbf{m}}$ has only real roots for each $\mathbf{m}\in\mathbb{P}^n$.

Let us consider the trivariate polynomials over generalized Stirling permutations

$$S_{\mathbf{m}}(x,y,z)=\sum_{\pi\in\mathcal{Q}_{\mathbf{m}}} x^{\mathsf{asc}(\pi)}y^{\mathsf{des}(\pi)}z^{\mathsf{plat}(\pi)}.$$

In order to state our first expansion formula for the partial γ-positivity of $S_{\mathbf{m}}(x,y,z)$, we need to introduce more statistics on multipermutations.

Definition 1.1. Let $\pi=\pi_1\pi_2\cdots\pi_m\in\mathfrak{S}_{\mathbf{m}}$, where $m=\sum_{i=1}^n m_i$. As usual, we set $\pi_0=\pi_{m+1}=-\infty$. A letter $k\in[n]$ is said to be *multiple* if $m_k>1$; and *single*, otherwise. An index $i\in[m]$ is called a *multiple* (resp. *single*) *descent* of π if $\pi_i>\pi_{i+1}$ and π_i is multiple (resp. single). A *double-ascent* (resp. *double-descent, peak, valley, ascent-plateau, descent-plateau*) of π is an index $i\in[m]$ such that $\pi_{i-1}<\pi_i<\pi_{i+1}$ (resp. $\pi_{i-1}>\pi_i>\pi_{i+1}$, $\pi_{i-1}<\pi_i>\pi_{i+1}$, $\pi_{i-1}>\pi_i<\pi_{i+1}$, $\pi_{i-1}<\pi_i=\pi_{i+1}$, $\pi_{i-1}>\pi_i=\pi_{i+1}$). It is clear that if i is a peak, then π_i must be single. We further distinguish a double-descent i to be single or multiple according to π_i is single or multiple. A descent-plateau i is *free* if there does not exist an integer ℓ, $1\le\ell<i$, such that $\pi_\ell=\pi_i$. A plateau i of π is said to be *unmovable* if i is neither a free descent-plateau nor an ascent-plateau. Let us introduce the statistics of π by

- $\mathsf{dasc}(\pi)=\#\{i\in[m]:\pi_{i-1}<\pi_i<\pi_{i+1}\}$, the number of **d**ouble-**asc**ents of π;
- $\mathsf{sddes}(\pi):=\#\{i\in[m]:\pi_{i-1}>\pi_i>\pi_{i+1}$ and π_i is single$\}$, the number of **s**ingle **d**ouble-**des**cents of π;
- $\mathsf{fdesp}(\pi):=\#\{i\in[m]:\pi_{i-1}>\pi_i=\pi_{i+1}$ and $\pi_\ell\neq\pi_i$ for all $\ell\in[i-1]\}$, the number of **f**ree **des**cent-**p**lateaus of π;
- $\mathsf{ascpp}(\pi):=\#\{i\in[m]:\pi_{i-1}<\pi_i=\pi_{i+1}$ or $\pi_{i-1}<\pi_i>\pi_{i+1}\}$, the number of **asc**ent-**p**lateaus and **p**eaks.
- $\mathsf{mdup}(\pi):=\#\{i\in[m]:$ i is a **m**ultiple **d**escent or a **u**nmovable **p**lateau$\}$.

For example, if $\pi = 15565333124411$, then $\mathsf{dasc}(\pi) = 2$, $\mathsf{sddes}(\pi) = 0$, $\mathsf{fdesp}(\pi) = 1$, $\mathsf{ascpp}(\pi) = 3$ and $\mathsf{mdup}(\pi) = 6$.

Now we are ready to state our first expansion formula for $S_{\mathbf{m}}(x, y, z)$.

Theorem 1.2. *For any $\mathbf{m} \in \mathbb{P}^n$ with $m_1 + m_2 + \cdots m_n = m$. The polynomial $S_{\mathbf{m}}(x, y, z)$ is partial γ-positive and has the expansion*

$$S_{\mathbf{m}}(x, y, z) = \sum_{i=0}^{m-n} z^i \sum_{j=1}^{\lfloor \frac{m+1-i}{2} \rfloor} \tilde{\gamma}_{\mathbf{m},i,j} (xy)^j (x+y)^{m+1-i-2j}, \tag{1.2}$$

where

$$\tilde{\gamma}_{\mathbf{m},i,j} = \#\{\pi \in \mathcal{Q}_{\mathbf{m}} : \mathsf{sddes}(\pi) = \mathsf{fdesp}(\pi) = 0, \mathsf{mdup}(\pi) = i, \mathsf{ascpp}(\pi) = j\}. \tag{1.3}$$

As will be seen in next section, Theorem 1.2 recovers two special cases for Stirling permutations due to Ma–Ma–Yeh (see Theorems 13 and 19 in [35]) and confirms their partial γ-positivity conjecture concerning the Jacobi–Stirling permutations (see Conjecture 2.1). The combinatorial proof of Theorem 1.2 uses Chen's context-free grammars and a group action on Stirling multipermutations that generalizes the original *Foata–Strehl group action* [21] on permutations.

For a multipermutation $\pi = \pi_1 \cdots \pi_m \in \mathfrak{S}_{\mathbf{m}}$, an index $i \in [m]$ is an *excedance* (resp. a *drop*, a *fixed point*) of π if $\pi_i > \sigma_i$ (resp. $\pi_i < \sigma_i$, $\pi_i = \sigma_i$), where the word $\sigma_1\sigma_2 \cdots \sigma_m$ is the nondecreasing rearrangement of π. For example, the multipermutation 213212, whose nondecreasing rearrangement is 112223, has two fixed points (indices 2 and 4), two excedances (indices 1 and 3) and two drops (indices 5 and 6). Excedances, drops and fixed points can be viewed as the cycle analog of descents, ascents and plateaux, respectively. A multipermutation in $\mathfrak{S}_{\mathbf{m}}$ is called a *Smirnov permutation* (resp. a *derangement*) if it has no plateaux (resp. fixed points). Using some formulas for the dimension of the homology of the Rees product of posets, Linusson, Shareshian and Wachs [33, Section 5] (see also [4, Theorem 2.25]) proved two combinatorial expansions for the γ-positivity of the descent polynomials on Smirnov permutations and the excedance polynomials on derangements. Our next goal is to put their results in the context of partial γ-positivity. We still need to introduce some particular multipermutation statistics before we can state our results.

Definition 1.3. Let $\pi = \pi_1\pi_2 \cdots \pi_m \in \mathfrak{S}_{\mathbf{m}}$ be a multipermutation. We can write π in the *compact form* $\hat{\pi}_1^{c_1}\hat{\pi}_2^{c_2} \cdots \hat{\pi}_k^{c_k}$ such that $|\pi| := \hat{\pi}_1\hat{\pi}_2 \cdots \hat{\pi}_k$ is a Smirnov permutation. For instance, if $\pi = 32122322111$, then its compact form is $3212^2 32^2 1^3$ and $|\pi| = 3212321$. An index $i \in [k]$ is a *compact double-descent* (resp. *compact double-ascent*) of π if i is a double-descent (resp. double-ascent) of $|\pi|$. A compact double-descent (resp. compact double-ascent) i of π is said to be *unmovable* if $\hat{\pi}_l = \hat{\pi}_i$, where l is the greatest (resp. smallest) index such that $1 \le l < i$ (resp. $i < l \le k$) and $\hat{\pi}_{l-1} < \hat{\pi}_i$ (resp. $\hat{\pi}_{l+1} < \hat{\pi}_i$) with the convention $\hat{\pi}_0 = \hat{\pi}_{k+1} = -\infty$; otherwise, index i is said to be *movable*. Continuing with the running example, the indices 2, 6 and 7 are compact double-descents and index 4 is a compact double-ascent, while only indices 4 and 6 are unmovable. Denote by $\mathsf{mdd}(\pi)/\mathsf{mda}(\pi)$ (resp. $\mathsf{udd}(\pi)/\mathsf{uda}(\pi)$) the number of **m**ovable (resp. **u**nmovable) compact **d**ouble-**d**escents/**d**ouble-**a**scents of π.

The following result generalizes the combinatorial expansion for the γ-positivity of the descent polynomials on Smirnov permutations proved in [33].

Theorem 1.4. *For any $\mathbf{m} \in \mathbb{P}^n$ with $m_1 + m_2 + \cdots m_n = m$, let*
$$A_{\mathbf{m}}(x,y,z) = \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\mathsf{asc}(\pi)} y^{\mathsf{des}(\pi)} z^{\mathsf{plat}(\pi)}.$$
The polynomial $A_{\mathbf{m}}(x,y,z)$ is partial γ-positive and has the expansion
$$A_{\mathbf{m}}(x,y,z) = \sum_{i=0}^{m-n} z^i \sum_{j=1}^{\lfloor \frac{m+1-i}{2} \rfloor} \gamma_{\mathbf{m},i,j} (xy)^j (x+y)^{m+1-i-2j},$$
where
$$\gamma_{\mathbf{m},i,j} = \#\{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathsf{mdd}(\pi) = 0, \mathsf{plat}(\pi) = i, \mathsf{des}(\pi) = j\}. \tag{1.4}$$

This theorem will be proved via introducing another generalization of the Foata–Strehl action on words (or permutations of a multiset). It turns out that our action also works for Stirling permutations, which results in a new partial γ-positivity expansion for $S_{\mathbf{m}}(x,y,z)$. Moreover, a restricted version of this action on words together with *Foata's first fundamental transformation* [34, Chapter 10.5] on words enable us to prove a partial γ-positivity expansion of
$$E_{\mathbf{m}}(x,y,z) := \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\mathsf{exc}(\pi)} y^{\mathsf{drop}(\pi)} z^{\mathsf{fix}(\pi)}, \tag{1.5}$$
where $\mathsf{exc}(\pi)$ (resp. $\mathsf{drop}(\pi)$, $\mathsf{fix}(\pi)$) denotes the number of excedances (resp. drops, fixed points) of π. To state our expansion, we need to introduce some necessary statistics on multipermutations.

Definition 1.5. Let $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_{\mathbf{m}}$ be a multipermutation. An index $i \in [m]$ is a *record* of π if $\pi_i \geq \pi_j$ for all $1 \leq j \leq i-1$. Suppose that $|\pi| = \hat{\pi}_1\hat{\pi}_2\cdots\hat{\pi}_k$ is the associated Smirnov permutation. An index i ($2 \leq i \leq k$) is called a *valid double-descent* (resp. *valid double-ascent*) of π if the following three conditions hold (with the convention $\hat{\pi}_{k+1} = \infty$):
- i is not a record of $|\pi|$;
- $\hat{\pi}_{i-1} > \hat{\pi}_i > \hat{\pi}_{i+1}$ (resp. $\hat{\pi}_{i-1} < \hat{\pi}_i < \hat{\pi}_{i+1}$);
- $\hat{\pi}_i \neq \hat{\pi}_l$, where l is the smallest (resp. greatest) index such that $i < l \leq k$ (resp. $2 \leq l < i$) and $\hat{\pi}_{l+1} > \hat{\pi}_i$ (resp. $\hat{\pi}_{l-1} > \hat{\pi}_i$).

An index i ($2 \leq i \leq k$) satisfies only the first two conditions above is called an *invalid double-descent* (resp. *invalid double-ascent*) of π. Denote by $\mathsf{vdd}(\pi)/\mathsf{vda}(\pi)$ the number of **v**alid **d**ouble-**d**escents/**d**ouble-**a**scents of π. An index $i \in [m]$ is called a *horizontal fixed point* of π if
- i is a consecutive record, i.e., both i and $i+1$ are records (with the convention that $m+1$ is always a record);
- or i is a plateau rather than a record.

Let $\mathsf{hfix}(\pi)$ be the number of horizontal fixed points of π. For example, if $\pi = 223212322134$, then $|\pi| = 2321232134$ and so $\mathsf{vdd}(\pi) = 1$ (index 7), while $\mathsf{hfix}(\pi) = 5$ (indices 1, 2, 8, 11 and 12). Note that $\mathsf{hfix}(\pi) \geq \mathsf{plat}(\pi) + 1$ for any $\pi \in \mathfrak{S}_{\mathbf{m}}$.

The following expansion for $E_{\mathbf{m}}(x,y,z)$ generalizes the combinatorial expansion for the γ-positivity of the excedance polynomials on multiset derangements proved in [33].

Theorem 1.6. *The polynomial $E_{\mathbf{m}}(x,y,z)$ is partial γ-positive and has the expansion*

$$E_{\mathbf{m}}(x,y,z)=\sum_{i=0}^{m} z^i \sum_{j=0}^{\lfloor\frac{m-i}{2}\rfloor} \bar{\gamma}_{\mathbf{m},i,j}(xy)^j(x+y)^{m-i-2j},$$

where

$$\bar{\gamma}_{\mathbf{m},i,j}=\#\{\pi\in\mathfrak{S}_{\mathbf{m}}:\mathsf{vdd}(\pi)=0,\mathsf{hfix}(\pi)=i,\mathsf{des}(\pi)=j+1\}. \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, we prove Ma–Ma–Yeh's partial γ-positivity conjecture for Jacobi–Stirling permutations from Theorem 1.2. In Section 3, we provide a proof of Theorem 1.2 by using the context-free grammars and a generalization of the Foata–Strehl action on Stirling permutations. In Section 4, an analytic approach to the partial γ-positivity of $S_{\mathbf{m}}(x,y,z)$ is studied by using the stable theory of multivariate polynomials developed by Borcea and Brändén. As a result, the descent polynomials over Stirling permutations with fixed number of plateaux are shown to be real-rooted. Theorems 1.4 and 1.6 are proved respectively in Sections 5 and 6 via introducing a word version of the Foata–Strehl group action on permutations. Finally, we conclude this paper with further remarks about Foata–Strehl group actions and open problems on γ-positive polynomials arising from enumerative combinatorics or geometry.

2. Jacobi–Stirling permutations

In this section, we show how Theorem 1.2 implies Ma–Ma–Yeh's partial γ-positivity conjecture for Jacobi–Stirling permutations.

The *Jacobi-Stirling numbers* $\mathrm{JS}(n,k;z)$, as a generalization of the Stirling number $S(n,k)$, were introduced in the study of a problem involving the spectral theory of powers of the classical second-order Jacobi differential expression (see [3]). Write the *Jacobi–Stirling polynomial* $\mathrm{JS}(n+k,k;z)$ as $p_{n,0}(k)+p_{n,1}(k)z+\cdots+p_{n,n}(k)z^n$. Generalizing the above study for $S(n+k,k)$, Gessel, Lin and Zeng [24] investigated the diagonal generating function

$$\sum_{k\geq 0} p_{n,i}(k)t^k=\frac{A_{n,i}(t)}{(1-t)^{3n+1-i}}$$

and showed that $A_{n,i}(t)$ is the descent polynomial over the Jacobi-Stirling permutations $\mathcal{JSP}_{n,i}$ defined in the same flavor as the Stirling permutations. Introduce the multiset

$$M_n:=\{1,1,\bar{1},2,2,\bar{2},\ldots,n,n,\bar{n}\},$$

where the elements are ordered by

$$\bar{1}<1<\bar{2}<2<\cdots<\bar{n}<n.$$

Let $[\bar{n}]:=\{\bar{1},\bar{2},\ldots,\bar{n}\}$. For any subset $S\subseteq[\bar{n}]$, we set $M_{n,S}=M_n\setminus S$. A permutation of $M_{n,S}$ is a *Jacobi-Stirling permutation* if for each $i\in[n]$, all entries between the two occurrences of i are larger than i. We denote by $\mathcal{JSP}_{n,S}$ the set of all Jacobi-Stirling permutations of $M_{n,S}$ and set

$$\mathcal{JSP}_{n,i} = \bigcup_{\substack{S\subseteq[\bar{n}]\\|S|=i}} \mathcal{JSP}_{n,S}.$$

Note that $\mathcal{JSP}_{n,n} = \mathcal{Q}_n$.

For any $n \geq 1$ and $0 \leq i \leq n$, let us consider the trivariate extension of $A_{n,i}(y)$:

$$\mathrm{JSP}_{n,i}(x,y,z) := \sum_{\pi\in\mathcal{JSP}_{n,i}} x^{\mathsf{asc}(\pi)} y^{\mathsf{des}(\pi)} z^{\mathsf{plat}(\pi)}.$$

Ma et al. [35] proved that $\mathrm{JSP}_{n,0}(x,y,z)$ and $\mathrm{JSP}_{n,0}(x,y,z)$ are all partial γ-positive and posed the conjecture that this phenomenon holds for all $\mathrm{JSP}_{n,i}(x,y,z)$.

Conjecture 2.1 (Ma, Ma and Yeh [35])**.** *For any $n \geq 1$ and $0 \leq i \leq n$, the polynomial $\mathrm{JSP}_{n,i}(x,y,z)$ is partial γ-positive.*

Proof. To see that Theorem 1.2 implies Conjecture 2.1, for any $S \subseteq [n]$ with $|S| = i$, define $\mathbf{m}(S) = (m_1, \ldots, m_{2n-i})$ where

$$m_\ell = \begin{cases} 2, & \text{if } \ell = p + |\{a \in [n] \setminus S : a \leq p\}| \text{ for some } 1 \leq p \leq n; \\ 1, & \text{otherwise.} \end{cases}$$

For instance, if $S = \{1, 2, 5, 7\} \subseteq [7]$, then $\mathbf{m}(S) = (2, 2, 1, 2, 1, 2, 2, 1, 2, 2)$. Let

$$\mathrm{JSP}_{n,S}(x,y,z) := \sum_{\pi\in\mathcal{JSP}_{n,S}} x^{\mathsf{asc}(\pi)} y^{\mathsf{des}(\pi)} z^{\mathsf{plat}(\pi)}.$$

It is routine to check that $\mathrm{JSP}_{n,S}(x,y,z) = S_{\mathbf{m}}(x,y,z)$ with $\mathbf{m} = \mathbf{m}(S)$. Therefore, by Theorem 1.2, $\mathrm{JSP}_{n,S}(x,y,z)$ is partial γ-positive, namely,

$$\mathrm{JSP}_{n,S}(x,y,z) = \sum_{k=0}^{3n-i-1} z^k \sum_{j=1}^{\lfloor\frac{3n-i+1-k}{2}\rfloor} a_{\mathbf{m},k,j}(xy)^j(x+y)^{3n-i+1-k-2j}.$$

Conjecture 2.1 then follows from $\mathrm{JSP}_{n,i}(x,y,z) = \sum_{\substack{S\in[n]\\|S|=i}} \mathrm{JSP}_{n,S}(x,y,z)$. $\square$

Remark 2.2. By the above discussion, any interpretation for the γ-coefficients $\tilde{\gamma}_{\mathbf{m},i,j}$ of $S_{\mathbf{m}}(x,y,z)$ also interprets $a_{\mathbf{m}(S),k,j}$, the γ-coefficients of $\mathrm{JSP}_{n,S}(x,y,z)$. Actually, the combinatorial interpretation for $\tilde{\gamma}_{\mathbf{m},i,j}$ in Theorem 1.2 specializes to the interpretations for the γ-coefficients of $\mathrm{JSP}_{n,0}(x,y,z)$ and $\mathrm{JSP}_{n,n}(x,y,z)$ found in [35, Theorems 13 and 19], which were proved by discussions based on the corresponding recurrences. Their approach seems hard to get our interpretation of $\tilde{\gamma}_{\mathbf{m},i,j}$ in Theorem 1.2.

Note that a second interpretation for $\tilde{\gamma}_{\mathbf{m},i,j}$ is provided in Corollary 5.2.

3. Proof of Theorem 1.2

3.1. **Context-free grammars and an equidistribution.** For a set $V = \{x, y, z, \ldots\}$ of commutative variables, a context-free grammar G is a set of substitution rules that replace a variable in V by a Laurent polynomial of variables in V. The formal derivative D associated with a context-free grammar G (introduced by Chen in [14]) is defined by $D(x) = G(x)$ for any $x \in V$ and satisfies the following relations:

$$D(u+v) = D(u) + D(v),$$
$$D(uv) = D(u)v + uD(v),$$

where u and v are two Laurent polynomials of variables in V. The context-free grammars have been found useful in studying various combinatorial structures [14–18,35,36], including permutations, increasing trees, labeled rooted trees and set partitions. For example, if $V = \{x, y\}$ and $G = \{x \to xy, y \to xy\}$, then $D(x) = xy$, $D^2(x) = xy(x+y)$ and $D^3(x) = D(xy)(x+y) + D(x+y)xy = x^3y + 4x^2y^2 + xy^3$. This is the grammar introduced by Dumont [18] to generate the bivariate Eulerian polynomials $D^n(x) = A_n(x, y)$.

Let $\pi = \pi_1\pi_2\cdots\pi_m \in \mathcal{Q}_{\mathbf{m}}$, where $m = \sum_{i=1}^n m_i$. A plateau i of π is called a *first plateaux* if $\pi_j \neq \pi_i$ for all $1 \leq j < i$. Let $\mathsf{fplat}(\pi)$ be the number of **f**irst **plat**eau of π. Denote by $\mathsf{sdes}(\pi)$ (resp. $\mathsf{mdes}(\pi)$) the number of **s**ingle (resp. **m**ultiple) **des**cents of π. We will apply the context-free grammars to prove the following equidistribution.

Lemma 3.1. *For any $\mathbf{m} \in \mathbb{P}^n$ with $m_1 + m_2 + \cdots + m_n = m$, the following two triplets of statistics are equidistributed on the Stirling permutations $\mathcal{Q}_{\mathbf{m}}$:*

$$(\mathsf{des}, \mathsf{plat}, \mathsf{asc}) \quad \textit{and} \quad (\mathsf{fplat} + \mathsf{sdes}, \mathsf{mdup}, \mathsf{asc}).$$

Proof. Note that i is a first plateau if and only if i is either a free descent-plateau or an ascent-plateau. In other words, a plateau is unmovable if and only if it is not a first plateaux. Thus, $\mathsf{plat}(\pi) - \mathsf{fplat}(\pi) = \mathsf{uplat}(\pi)$, the number of **u**nmovable **plat**eau of π. We aim to show that the quintuplets

$$(\mathsf{sdes}, \mathsf{mdes}, \mathsf{fplat}, \mathsf{uplat}, \mathsf{asc}) \quad \text{and} \quad (\mathsf{sdes}, \mathsf{fplat}, \mathsf{mdes}, \mathsf{uplat}, \mathsf{asc}) \tag{3.1}$$

are equidistributed on $\mathcal{Q}_{\mathbf{m}}$, from which the lemma follows.

For a Stirling permutation $\pi \in \mathcal{Q}_{\mathbf{m}}$, we first introduce a grammatical labeling of π as follows:

- (L_1) If i is a single descent, then put a superscript label x right after π_i;
- (L_2) If i is a multiple descent, then put a superscript label $\tilde{x}$ right after π_i;
- (L_3) If i is a first plateau, then put a superscript label $\tilde{y}$ right after π_i;
- (L_4) If i is a unmovable plateau, then put a superscript label y right after π_i;
- (L_5) If i is an ascent, then put a superscript label z right after π_i.

Recall that we always set $\pi_0 = \pi_{m+1} = 0$. For example, if $\pi = 15565333124411$, then the labeling of π is

$$0^z1^z5^{\tilde{y}}5^z6^x5^{\tilde{x}}3^{\tilde{y}}3^y3^{\tilde{x}}1^z2^z4^{\tilde{y}}4^{\tilde{x}}1^y1^{\tilde{x}}0.$$

It is clear that the weight $x^{\mathsf{sdes}(\pi)}\tilde{x}^{\mathsf{mdes}(\pi)}\tilde{y}^{\mathsf{fplat}(\pi)}y^{\mathsf{uplat}(\pi)}z^{\mathsf{asc}(\pi)}$ is the product of all the superscripts in the labelings of π. Let $V = \{x, \tilde{x}, y, \tilde{y}, z\}$. For integer $k \geq 2$, introduce the context-free grammar

$$G_k = \{x \to \tilde{x}\tilde{y}y^{k-2}z, \tilde{x} \to \tilde{x}\tilde{y}y^{k-2}z, y \to \tilde{x}\tilde{y}y^{k-2}z, \tilde{y} \to \tilde{x}\tilde{y}y^{k-2}z, z \to \tilde{x}\tilde{y}y^{k-2}z\}.$$

Also, define the context-free grammar

$$G_1 = \{x \to xz, \tilde{x} \to xz, y \to xz, \tilde{y} \to xz, z \to xz\}.$$

For any $k \geq 1$, let D_k be the formal derivative associated with the context-free grammar G_k. We claim that

$$D_{m_n} D_{m_{n-1}} \cdots D_{m_1}(z) = \sum_{\pi \in \mathcal{Q}_{\mathbf{m}}} x^{\mathsf{sdes}(\pi)} \tilde{x}^{\mathsf{mdes}(\pi)} \tilde{y}^{\mathsf{fplat}(\pi)} y^{\mathsf{uplat}(\pi)} z^{\mathsf{asc}(\pi)}.$$

The equidistribution (3.1) then follows from this claim and the fact that the context-free grammars G_k are symmetric in $\tilde{x}$ and $\tilde{y}$ for all $k \geq 1$.

It remains to show the claim. We proceed by induction on n. The statement is obviously true for the initial case $n = 0$, as $\mathcal{Q}_0 = \{0^z 0\}$. Suppose that we have all labeled permutations in $\mathcal{Q}_{\mathbf{m}'}$, where $\mathbf{m}' = (m_1, m_2, \ldots, m_{n-1})$. Note that every permutation in $\mathcal{Q}_{\mathbf{m}}$ can be constructed from a permutation $\sigma \in \mathcal{Q}_{\mathbf{m}'}$ by inserting n^{m_n}, m_n copies of n, to the position between σ_i and σ_{i+1} for some nonnegative integer i. The changes of labelings are illustrated as follows for any $v \in \{x, \tilde{x}, y, \tilde{y}, z\}$ (no matter what the label v is):

- If $m_n \geq 2$, then
$$\cdots \sigma_i^v \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^z n^{\tilde{y}} n^y \cdots n^y n^{\tilde{x}} \sigma_{i+1} \cdots ,$$
where n^y appears $(m_n - 2)$ times.
- Otherwise $m_n = 1$, and
$$\cdots \sigma_i^v \sigma_{i+1} \cdots \mapsto \cdots \sigma_i^z n^x \sigma_{i+1} \cdots .$$

In either case, the insertion of n^{m_n} corresponds to one substitution rule in G_{m_n}. Since the action of D_{m_n} on elements of $\mathcal{Q}_{\mathbf{m}'}$ generates all elements of $\mathcal{Q}_{\mathbf{m}}$, the claim holds. This completes the proof of the lemma. $\square$

In order to finish the proof of Theorem 1.2, we need a generalization of the Foata–Strehl-action for Stirling permutations that will be introduced below.

3.2. **Generalized Foata–Strehl actions on Stirling permutations.** For a Stirling permutation $\pi \in \mathcal{Q}_{\mathbf{m}}$ and a value $x \in [n]$, suppose that π_ℓ is the leftmost occurrence of x in π. We call x a *free descent-plateau value* (resp. a *single double descents value*, a *double ascents value*) of π if ℓ is a free descent plateau (resp. a single double descents, a double ascents) of π. Then we introduce the *Generalized Foata-Strehl action* (GFS-action for short) $\tilde{\varphi}_x$ as follows:

- If x is a free descent-plateau value or a single double descents value of π, then $\tilde{\varphi}_x(\pi)$ is obtained from π by moving π_ℓ to the right of the letter π_k, where $k = \max\{a : 0 \leq a \leq \ell - 2, \pi_a < x\}$;
- If x is a double ascents value of π, then $\tilde{\varphi}_x(\pi)$ is obtained from π by moving π_ℓ to the left of the letter π_k, where $k = \min\{a : \ell + 2 \leq a \leq m + 1, \pi_a \leq x\}$;
- If x is not in the above two cases, then let $\tilde{\varphi}_x(\pi) = \pi$.

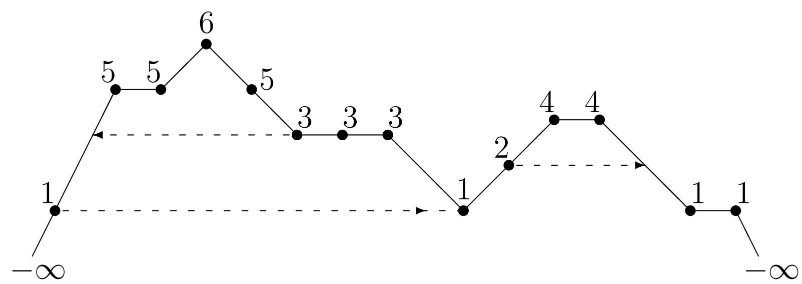

FIGURE 1. GFS-actions on Stirling permutation 15565333124411

The GFS-action has a nice visualization as depicted in Fig. 1. For example, if $\pi = 15565333124411$, then $\tilde{\varphi}_1(\pi) = 5565333\mathbf{1}124411$, $\tilde{\varphi}_3(\pi) = 1\mathbf{3}556533124411$ and $\tilde{\varphi}_2(\pi) = 155653331442\mathbf{2}11$. When $m_1 = m_2 = \cdots = m_n = 1$, the GFS-action becomes the version of the *Modified Foata-Strehl action* introduced in [29].

We are now ready for the proof of Theorem 1.2.

Proof of Theorem 1.2. By Lemma 3.1, we have

$$S_{\mathbf{m}}(x,y,z) = \sum_{\pi \in \mathcal{Q}_{\mathbf{m}}} x^{\mathsf{asc}(\pi)} y^{\mathsf{fplat}(\pi)+\mathsf{sdes}(\pi)} z^{\mathsf{mdup}(\pi)}.$$

Let us define the set $\mathcal{Q}_{\mathbf{m},i} = \{\pi \in \mathcal{Q}_{\mathbf{m}} : \mathsf{mdup}(\pi) = i\}$. Then Theorem 1.2 is equivalent to

$$\sum_{\pi \in \mathcal{Q}_{\mathbf{m},i}} x^{\mathsf{asc}(\pi)} y^{\mathsf{fplat}(\pi)+\mathsf{sdes}(\pi)} = \sum_{\pi \in \widetilde{\mathcal{Q}_{\mathbf{m},i}}} (xy)^{\mathsf{ascpp}(\pi)} (x+y)^{m+1-i-2\times\mathsf{ascpp}(\pi)}, \tag{3.2}$$

where $\widetilde{\mathcal{Q}_{\mathbf{m},i}} := \{\pi \in \mathcal{Q}_{\mathbf{m},i} : \mathsf{sddes}(\pi) = \mathsf{fdesp}(\pi) = 0\}$.

Clearly, the GFS-actions $\tilde{\varphi}_x$'s are involutions and commute. Thus, for any $S \subseteq [n]$ we can define the function $\tilde{\varphi}_S : \mathcal{Q}_{\mathbf{m}} \to \mathcal{Q}_{\mathbf{m}}$ by $\tilde{\varphi}_S = \prod_{x \in S} \tilde{\varphi}_x$, where the product is the functional compositions. For instance, continuing with the example in Fig. 1, we have $\tilde{\varphi}_{\{1,3\}}(\pi) = \mathbf{3}5565333\mathbf{1}124411$. Hence the group $\mathbb{Z}_2^n$ acts on $\mathcal{Q}_{\mathbf{m}}$ via the function $\tilde{\varphi}_S$. Since the statistic "mdup" is invariant under this group action, it divides $\mathcal{Q}_{\mathbf{m},i}$ into some disjoint orbits. For each $\pi \in \mathcal{Q}_{\mathbf{m},i}$, let $\mathrm{Orb}(\pi) = \{g(\pi) : g \in \mathbb{Z}_2^n\}$ be the orbit of π under the GFS-action. Note that x is a free descent-plateau value or a single double descents value of π if and only if x is a double ascents value of $\tilde{\varphi}_x(\pi)$. Thus, there exists a unique Stirling permutation $\tilde{\pi}$ in $\mathrm{Orb}(\pi)$ such that $\mathsf{sddes}(\tilde{\pi}) = \mathsf{fdesp}(\tilde{\pi}) = 0$, that is, $\widetilde{\mathcal{Q}_{\mathbf{m},i}} \bigcap \mathrm{Orb}(\pi) = \{\tilde{\pi}\}$. Therefore, we have

$$\begin{aligned}\sum_{\sigma \in \mathrm{Orb}(\pi)} x^{\mathsf{asc}(\sigma)} y^{\mathsf{fplat}(\sigma)+\mathsf{sdes}(\sigma)} &= x^{\mathsf{asc}(\tilde{\pi})-\mathsf{dasc}(\tilde{\pi})} y^{\mathsf{fplat}(\tilde{\pi})+\mathsf{sdes}(\tilde{\pi})} (x+y)^{\mathsf{dasc}(\tilde{\pi})} \\ &= (xy)^{\mathsf{ascpp}(\tilde{\pi})} (x+y)^{m+1-i-2\times\mathsf{ascpp}(\tilde{\pi})},\end{aligned}$$

where the second equality follows from the following relationships

$$\mathsf{asc}(\tilde{\pi}) - \mathsf{dasc}(\tilde{\pi}) = \mathsf{fplat}(\tilde{\pi}) + \mathsf{sdes}(\tilde{\pi}) = \mathsf{ascpp}(\tilde{\pi}) \tag{3.3}$$

and

$$\mathsf{dasc}(\tilde{\pi}) = m + 1 - i - 2 \times \mathsf{ascpp}(\tilde{\pi}). \tag{3.4}$$

Summing over all orbits of $\mathcal{Q}_{\mathbf{m},i}$ under the GFS-action then gives (3.2).

It remains to show the above relationships. Since $\tilde{\pi}$ has neither free descent-plateau nor single double descents, every first plateau must be an ascent-plateau and every single descent is a peak. Thus, we have $\mathsf{fplat}(\tilde{\pi}) + \mathsf{sdes}(\tilde{\pi}) = \mathsf{ascpp}(\tilde{\pi})$. As each ascent is followed immediately by an ascent, a plateau or a descent, we have $\mathsf{asc}(\tilde{\pi}) = \mathsf{dasc}(\tilde{\pi}) + \mathsf{ascpp}(\tilde{\pi})$, which proves (3.3). Clearly, we have $\mathsf{asc}(\tilde{\pi}) + \mathsf{plat}(\tilde{\pi}) + \mathsf{des}(\tilde{\pi}) = m + 1$ and $\mathsf{mdup}(\tilde{\pi}) + \mathsf{fplat}(\tilde{\pi}) + \mathsf{sdes}(\tilde{\pi}) = \mathsf{plat}(\tilde{\pi}) + \mathsf{des}(\tilde{\pi})$. It then follows that $\mathsf{mdup}(\tilde{\pi}) + \mathsf{asc}(\tilde{\pi}) + \mathsf{fplat}(\tilde{\pi}) + \mathsf{sdes}(\tilde{\pi}) = m + 1$ and so by (3.3),

$$\mathsf{dasc}(\tilde{\pi}) = \mathsf{asc}(\tilde{\pi}) - \mathsf{ascpp}(\tilde{\pi}) = m + 1 - \mathsf{mdup}(\tilde{\pi}) - 2 \times \mathsf{ascpp}(\tilde{\pi}),$$

which is relationship (3.4). This completes the proof of the theorem. □

4. Stability

In this section, we study the analytic properties of the trivariate polynomial $S_{\mathbf{m}}(x, y, z)$. From the involution

$$\pi_1\pi_2\cdots\pi_m \mapsto \pi_m\pi_{m-1}\cdots\pi_1,$$

we know that $S_{\mathbf{m}}(x, y, z)$ is symmetric in x and y. Hence, the partial γ-positivity of $S_{\mathbf{m}}(x, y, z)$ is equivalent to the γ-positivity of the following refined descent polynomials

$$S_{\mathbf{m},i}(x) = \sum_{\pi} x^{\mathsf{des}(\pi)},$$

where the sum runs over all Stirling permutations $\pi \in \mathcal{Q}_{\mathbf{m}}$ with $\mathsf{plat}(\pi) = i$. It is known (see [10, Remark 7.3.1]) that the real-rootedness of a polynomial with symmetric coefficients implies the γ-positivity of such polynomial. This motivates us to study an alternative approach to the partial γ-positivity of $S_{\mathbf{m}}(x, y, z)$.

Let us recall the stability of multivariate polynomials, which has been developed enormously by Borcea and Brändén [5,6]. A polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is said to be *stable* if either $f(x_1, \ldots, x_n) \neq 0$ whenever $\mathrm{Im}(x_i) > 0$ for all i or f is identically zero. Note that a univariate real polynomial is stable if and only if it has only real roots. Several multivariate Eulerian polynomials have been shown to be stable, see [8, 11, 16, 43, 45].

Our main result of this section is stated as follows.

Theorem 4.1. *Let $p(x, y, z)$ be a trivariate homogeneous polynomial with nonnegative coefficients. If $p(x, y, z)$ is stable and symmetric in x and y, then $p(x, y, z)$ is partial γ-positive.*

In order to prove Theorem 4.1, we shall introduce some basic results about stability-preserving linear operators.

Lemma 4.2 (See [44, Lemma 2.4]). *Given $i, j \in [n]$, the following operations preserve stability of $f \in \mathbb{R}[x_1, \ldots, x_n]$:*

(1) Differentiation: $f \mapsto \partial f/\partial x_i$.
(2) Diagonalization: $f \mapsto f|_{x_i = x_j}$.

(3) Specialization: *for* $a \in \mathbb{R}$, $f \mapsto f|_{x_i=a}$.

Now it is time for us to prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that $p(x,y,z) = \sum_{i=0}^{d} s_i(x,y)z^i$. We first prove that for each $0 \le k \le d$, the polynomial $s_k(x,y)$ is stable. By taking the k-th order partial derivative with respect to z of the real stable polynomial $p(x,y,z)$ it follows from Lemma 4.2 that $\sum_{i=k}^{d}(i)_k s_i(x,y)z^{i-k}$ is stable, where $(i)_k = i(i-1)\cdots(i-k+1)$. Note that if z is in the upper half plane then so is $-1/z$. Hence, we obtain the stability of $\sum_{i=k}^{d}(-1)^{i-k}(i)_k s_i(x,y)z^{d-i}$. Similarly, by taking the $(d-k)$-th order partial derivative with respect to z, we get the stability of $s_k(x,y)$.

We next prove that $s_k(x,y)$ is homogeneous γ-positive. By Lemma 4.2, we get that $s_k(x,1)$ is real-rooted. Since $s_k(x,y)$ is symmetric in x and y, we know that the coefficients of $s_k(x,1)$ are symmetric. Hence, the polynomial $s_k(x,1)$ is γ-positive. Since $s_k(x,y)$ is homogeneous in x and y, we obtain the homogeneous γ-positivity of $s_k(x,y)$. This completes the proof. $\square$

We proceed to use Theorem 4.1 to prove the partial γ-positivity of $S_{\mathbf{m}}(x,y,z)$. In order to do this, it suffices to show that $S_{\mathbf{m}}(x,y,z)$ is stable. For an element $\pi \in \mathcal{Q}_{\mathbf{m}}$, let $\mathcal{D}(\pi)$ and $\mathcal{A}(\pi)$ be the set of descents and ascents of π, respectively. If $\kappa = \max_i m_i$ and $1 \le j < \kappa$, define $\mathcal{P}_j(\pi)$ to be the set of indices i such that $\pi_i = \pi_{i+1}$ where $\pi_1 \cdots \pi_{i-1}$ contains $j-1$ instances of π_i. Haglund and Visontai [26, Theorem 3.5] showed that the multivariate polynomial

$$\sum_{\pi \in \mathcal{Q}_{\mathbf{m}}} \prod_{i \in \mathcal{D}(\pi)} x_{\pi_i} \prod_{i \in \mathcal{A}(\pi)} y_{\pi_i} \prod_{j=1}^{\kappa-1}\Big(\prod_{i \in \mathcal{P}_j(\pi)} z_{j,\pi_i}\Big)$$

is stable. By diagonalizing the variables x_{π_i}, y_{π_i} and z_{j,π_i} to x, y and z respectively, it follows from Lemma 4.2 that the polynomial $S_{\mathbf{m}}(x,y,z)$ is stable. As an application of this result we get the real-rootedness of $S_{\mathbf{m},i}(x)$.

Corollary 4.3. *For any* $\mathbf{m} \in \mathbb{P}^n$ *with* $m_1 + m_2 + \cdots m_n = m$ *and any* $0 \le i \le m-1$, *the polynomial* $S_{\mathbf{m},i}(x)$ *has only real roots.*

5. Foata–Strehl actions on words and proof of Theorem 1.4

In this section, we introduce another generalization of Foata–Strehl actions on words and give a proof of Theorem 1.4.

Let $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_{\mathbf{m}}$ be a multipermutation whose compact form is $\hat{\pi}_1^{c_1}\hat{\pi}_2^{c_2}\cdots\hat{\pi}_k^{c_k}$. Thus, its associated Smirnov permutation is $|\pi| = \hat{\pi}_1\hat{\pi}_2\cdots\hat{\pi}_k$. For $x \in [k]$, we introduce the *word version of Foata–Strehl action* (abbreviated as *WFS-action*) φ_x as follows:

- If x is a double-descent of $|\pi|$, then $\varphi_x(\pi)$ is obtained from π by moving $\hat{\pi}_x^{c_x}$ to the right of the letter π_ℓ, where $\ell = \max\{a : 0 \le a \le \sum_{i=1}^{x-1} c_i \text{ and } \pi_a < \hat{\pi}_x\}$;
- If x is a double-ascent of $|\pi|$, then $\varphi_x(\pi)$ is obtained from π by moving $\hat{\pi}_x^{c_x}$ to the left of the letter π_ℓ, where $\ell = \min\{a : \sum_{i=1}^{x} c_i < a \le k \text{ and } \pi_a < \hat{\pi}_x\}$;
- If x is neither a double-descent nor a double-ascent, then let $\varphi_x(\pi) = \pi$.

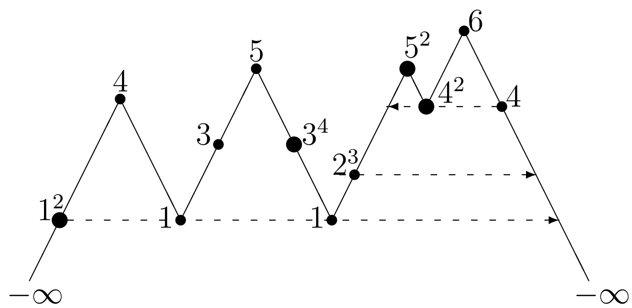

FIGURE 2. Modified WFS-actions on $1^2 41353^4 12^3 5^2 4^2 64$

For instance, if $\pi = 1^2 41353^4 1 2^3 5^2 4^2 6 4$, then $|\pi| = 141353125464$ and

$$\varphi_8(\pi) = 1^2 41353^4 15^2 4^2 64 2^3, \quad \varphi_{12}(\pi) = 1^2 41353^4 12^3 4 5^2 4^2 6.$$

Proof of Theorem 1.4. Let $\mathfrak{S}_{\mathbf{m},i} := \{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathsf{plat}(\pi) = i\}$. For each $\pi \in \mathfrak{S}_{\mathbf{m},i}$, if its associated Smirnov permutation $|\pi|$ has length k, then $i + k = m$. For each $x \in [k]$, let us introduce the modified WFS-action φ'_x by

$$\varphi'_x(\pi) = \begin{cases} \varphi_x(\pi), & \text{if } x \text{ is a movable double-descent/double-ascent of } \pi; \\ \pi, & \text{otherwise.} \end{cases}$$

See Fig. 2 for a nice visualization of the modified WFS-actions on $\pi = 1^2 41353^4 12^3 5^2 4^2 64$. Clearly, the action φ'_x preserves the number of plateau, thus $\varphi'_x(\pi) \in \mathfrak{S}_{\mathbf{m},i}$.

If x is a movable double-descent/double-ascent of $|\pi|$, then the pack of letters $\hat{\pi}_x^{c_x}$ is called a *(movable) double-descent/double-ascent pack* of π. We call two elements in $\mathfrak{S}_{\mathbf{m},i}$ *equivalent* if one can be obtained from the other by a sequence of actions of the form $\pi \mapsto \varphi'_x(\pi)$. This defines an equivalence relation on $\mathfrak{S}_{\mathbf{m},i}$, since a pack $\hat{\pi}_x^{c_x}$ is a double-descent pack of π if and only if it is a double-ascent pack of $\varphi'_x(\pi)$. Moreover, the equivalence class containing π, denoted $\mathrm{Orb}(\pi)$, has a unique permutation $\tilde{\pi}$ such that $\mathsf{mdd}(\tilde{\pi}) = 0$, the one with least descents. Therefore,

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} x^{\mathsf{asc}(\sigma)} y^{\mathsf{des}(\sigma)} = (xy)^{\mathsf{des}(\tilde{\pi})}(x+y)^{\mathsf{mda}(\tilde{\pi})} = (xy)^{\mathsf{des}(\tilde{\pi})}(x+y)^{m+1-i-2\times\mathsf{des}(\tilde{\pi})}, \tag{5.1}$$

where the second equality follows from the relationship

$$\mathsf{mda}(\tilde{\pi}) = m + 1 - i - 2 \times \mathsf{des}(\tilde{\pi}). \tag{5.2}$$

It suffices to prove the above relationship for $\tilde{\pi}$ being a Smirnov word. For such a $\tilde{\pi}$, Eq. (5.2) is a consequence of the following simple facts:

$$\mathsf{mdd}(\tilde{\pi}) + \mathsf{mda}(\tilde{\pi}) + \mathsf{udd}(\tilde{\pi}) + \mathsf{uda}(\tilde{\pi}) + \mathsf{peak}(\tilde{\pi}) + \mathsf{val}(\tilde{\pi}) = m,$$
$$\mathsf{udd}(\tilde{\pi}) + \mathsf{peak}(\tilde{\pi}) = \mathsf{uda}(\tilde{\pi}) + \mathsf{val}(\tilde{\pi}) + 1 = \mathsf{des}(\tilde{\pi}) \quad \text{and} \quad \mathsf{mdd}(\tilde{\pi}) = 0,$$

where $\mathsf{peak}(\tilde{\pi})$ and $\mathsf{val}(\tilde{\pi})$ denote respectively the number of peaks and valleys of $\tilde{\pi}$.

Let $\widetilde{\mathfrak{S}_{\mathbf{m},i}} := \{\tilde{\pi} \in \mathfrak{S}_{\mathbf{m},i} : \mathsf{mdd}(\tilde{\pi}) = 0\}$. Using (5.1) and summing over all equivalence classes of $\mathfrak{S}_{\mathbf{m},i}$ yields

$$\sum_{\pi \in \mathfrak{S}_{\mathbf{m},i}} x^{\mathsf{asc}(\pi)} y^{\mathsf{des}(\pi)} = \sum_{\tilde{\pi} \in \widetilde{\mathfrak{S}_{\mathbf{m},i}}} (xy)^{\mathsf{des}(\tilde{\pi})} (x+y)^{m+1-i-2\times \mathsf{des}(\tilde{\pi})},$$

which is equivalent to (1.4). □

Lemma 5.1. *The set $\mathcal{Q}_{\mathbf{m}}$ is invariant under the modified WFS-actions.*

Proof. For any $\pi \in \mathcal{Q}_{\mathbf{m}}$, either $\varphi'_x(\pi) = \pi$ or $\varphi'_x(\pi)$ is obtained from π by moving copies of letters across some letters not smaller than them, $\varphi'_x(\pi) \in \mathcal{Q}_{\mathbf{m}}$. □

Lemma 5.1 and Eq. (5.1) lead to a second interpretation for the γ-coefficients $\tilde{\gamma}_{\mathbf{m},i,j}$, which is new even when $\mathbf{m} = \{1, 1, 2, 2, \ldots, n, n\}$.

Corollary 5.2. *The γ-coefficients $\tilde{\gamma}_{\mathbf{m},i,j}$ defined in 1.2 has another interpretation*

$$\tilde{\gamma}_{\mathbf{m},i,j} = \#\{\pi \in \mathcal{Q}_{\mathbf{m}} : \mathsf{mdd}(\pi) = 0, \mathsf{plat}(\pi) = i, \mathsf{des}(\pi) = j\}. \tag{5.3}$$

Let $\pi = \pi_1\pi_2\cdots\pi_m \in \mathfrak{S}_{\mathbf{m}}$ be a multipermutation with $\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathbb{P}^n$. Define the *complement* π^c of π by

$$\pi^c = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_m).$$

Let $\mathrm{Wasc}(\pi)$ be the set of indices $i \in [m]$ such that $\pi_i \le \pi_{i+1}$ with the convention $\pi_{m+1} = +\infty$. For $\Omega \subseteq \mathbb{Z}$, we denote by $\mathrm{Stab}(\Omega)$ the set of all subsets of Ω which do not contain two consecutive integers. For each $j \ge 1$, let

$$W_{\mathbf{m},j} := \{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathrm{Wasc}(\pi) \in \mathrm{Stab}([n]) \text{ and } |\mathrm{Wasc}(\pi)| = j\}.$$

The following alternative description of the γ-coefficients $\gamma_{\mathbf{m},0,j}$ proved in [33, Section 5] can be deduced from Theorem 1.4.

Theorem 5.3 (Linusson, Shareshian and Wachs)**.** *For any $\mathbf{m} \in \mathbb{P}^n$ and $j \ge 1$,*

$$\gamma_{\mathbf{m},0,j} = |W_{\mathbf{m},j}|.$$

We will also need the following multiplicity changing bijection θ constructed by Han [27, Section 4].

Lemma 5.4. *Let $\mathbf{m} = (m_1, m_2, \cdots, m_n) \in \mathbb{P}^n$ and $\mathbf{m}' = (m'_1, m'_2, \cdots, m'_n)$ a rearrangement of $\mathbf{m}$. There exists a bijection $\theta : \mathfrak{S}_{\mathbf{m}} \to \mathfrak{S}_{\mathbf{m}'}$ such that for each $\pi \in \mathfrak{S}_{\mathbf{m}}$,*

$$\mathrm{Wasc}(\pi) = \mathrm{Wasc}(\theta(\pi)).$$

Proof of Theorem 5.3. Let $\mathrm{Wdes}(\pi)$ be the set of indices $i \in [m]$ such that $\pi_i \ge \pi_{i+1}$ with the convention $\pi_{m+1} = -\infty$. For each $j \ge 1$, set

$$W'_{\mathbf{m},j} := \{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathrm{Wdes}(\pi) \in \mathrm{Stab}([n]) \text{ and } |\mathrm{Wdes}(\pi)| = j\}$$

and

$$\Gamma_{\mathbf{m},j} := \{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathsf{mdd}(\pi) = \mathsf{plat}(\pi) = 0, \mathsf{des}(\pi) = j\}.$$

A WFS-action $\pi \mapsto \varphi_x(\pi)$ is called an *unmovable action* on π if x is an unmovable double descent of π. Applying all possible unmovable actions to an element $\pi \in \Gamma_{\mathbf{m},j}$ results in a permutation $f(\pi) \in W'_{\mathbf{m},j}$. The mapping $\pi \mapsto f(\pi)$ is a bijection between $\Gamma_{\mathbf{m},j}$ and $W'_{\mathbf{m},j}$. On the other hand, the mapping $\pi \mapsto \pi^c$ sets up an one-to-one correspondence between $W'_{\mathbf{m},j}$ and $W_{\overline{\mathbf{m}},j}$, where $\overline{\mathbf{m}} = (m_n, m_{n-1}, \ldots, m_1)$. By Lemma 5.4, θ is a bijection between $W_{\overline{\mathbf{m}},j}$ and $W_{\mathbf{m},j}$. Thus, $|\Gamma_{\mathbf{m},j}| = |W_{\mathbf{m},j}|$. By Theorem 1.4, $\gamma_{\mathbf{m},0,j} = |\Gamma_{\mathbf{m},j}|$ and the desired result follows. □

6. Restricted Foata–Strehl action and proof of Theorem 1.6

This section is devoted to the proof of Theorem 1.6. We will apply the so-called *Foata's first fundamental transformation* $\pi \mapsto \breve{\pi}$ on $\mathfrak{S}_{\mathbf{m}}$ (see [34, pp. 197-199]). It is convenience to recall this transformation by means of one example. Consider the multipermutation

$$\pi = 3112364222665175.$$

Its *increasing factorization* (see [34, Lemma 10.2.1]) is

$$(3112, 3, 64222, 6, 651, 75),$$

where each component is a *dominated word*, i.e., a maximal word whose first letter is strictly greater than the others. Then form the two rows of words

$$\Delta(\pi) = \begin{bmatrix} 1&1&2&3&3&4&2&2&2&6&6&5&1&6&5&7 \\ 3&1&1&2&3&6&4&2&2&2&6&6&5&1&7&5 \end{bmatrix},$$

where the bottom row is π and the top row is the concatenation of all shifted components of π. Here a *shifted component* is obtained from a component by shifting its first letter to the end. For instance, the shifted component of 3112 is 1123. Finally, reshuffling all the columns of $\Delta(\pi)$ so that the top row is in increasing order

$$\begin{bmatrix} 1&1&1&2&2&2&2&3&3&4&5&5&6&6&6&7 \\ 3&1&5&1&4&2&2&2&3&6&6&7&2&6&1&5 \end{bmatrix}.$$

The word $\breve{\pi} = 3151422236672615$ is then defined to be the bottom row above.

The following lemma is a direct consequence of Foata's first fundamental transformation $\pi \mapsto \breve{\pi}$ on $\mathfrak{S}_{\mathbf{m}}$.

Lemma 6.1. *The mapping $\pi \mapsto \breve{\pi}$ is a bijection of $\mathfrak{S}_{\mathbf{m}}$ onto itself that transforms the pair* $(\mathsf{hfix}, \mathsf{des} - 1)$ *to* $(\mathsf{fix}, \mathsf{exc})$.

Proof. The fact that $\pi \mapsto \breve{\pi}$ is a bijection of $\mathfrak{S}_{\mathbf{m}}$ onto itself that transforms des to exc was proved in [34, Theorem 10.5.2]. It is evident from the construction above that $\mathsf{hfix}(\pi) = \mathsf{fix}(\breve{\pi})$ since i is a horizontal fixed point of w if and only if the ith column of $\Delta(\pi)$ is $\binom{\pi_i}{\pi_i}$, which corresponds to a fixed point of $\breve{\pi}$. □

It follows from Lemma 6.1 and the definition of $E_{\mathbf{m}}(x, y, z)$ in 1.5 that

$$E_{\mathbf{m}}(x, y, z) = \sum_{\pi \in \mathfrak{S}_{\mathbf{m}}} x^{\mathsf{des}(\pi)-1} y^{m+1-\mathsf{des}(\pi)-\mathsf{hfix}(\pi)} z^{\mathsf{hfix}(\pi)} \tag{6.1}$$

since $\mathsf{fix}(\pi) + \mathsf{exc}(\pi) + \mathsf{drop}(\pi) = m$ for each $\pi \in \mathfrak{S}_{\mathbf{m}}$. Now we proceed to prove Theorem 1.6 via introducing a restricted version of the Foata–Strehl action on words.

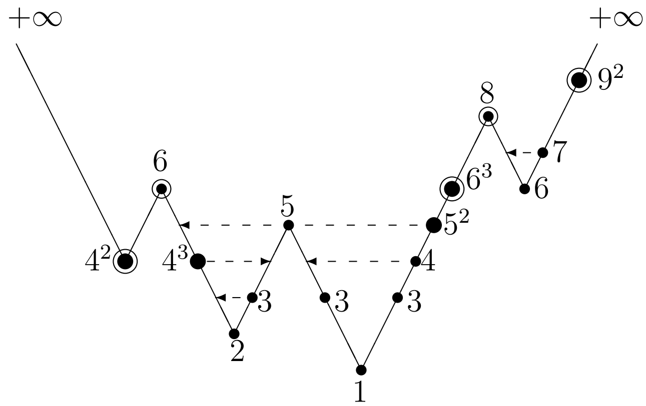

FIGURE 3. Restricted WFS-actions on $4^2 6 4^3 2 3 5 5 3 1 3 4 5^2 6^3 8 6 7 9^2$ (all record values are circled)

Let $\pi \in \mathfrak{S}_{\mathbf{m}}$ with $|\pi| = \hat{\pi}_1 \hat{\pi}_2 \cdots \hat{\pi}_k$. For any $x \in [k]$, let us introduce the *complement* $\bar{\varphi}_x$ of the WFS-action φ_x by

$$\bar{\varphi}_x(\pi) = (\varphi_x(\pi^c))^c.$$

Define the restricted WFS-action $\bar{\varphi}'_x$ by

$$\bar{\varphi}'_x(\pi) = \begin{cases} \bar{\varphi}_x(\pi), & \text{if } x \text{ is a valid double-descent/double-ascent of } \pi; \\ \pi, & \text{otherwise.} \end{cases}$$

See Fig. 2 for a visualization of the restricted WFS-actions on $4^2 6 4^3 2 3 5 5 3 1 3 4 5^2 6^3 8 6 7 9^2$.

Proof of Theorem 1.6. It is evident from the definition of $\bar{\varphi}'_x$ that it preserves the number of horizontal fixed points. Let $\mathfrak{S}'_{\mathbf{m},i} := \{\pi \in \mathfrak{S}_{\mathbf{m}} : \mathsf{hfix}(\pi) = i\}$. Let $\pi \in \mathfrak{S}'_{\mathbf{m},i}$ with compact form $\hat{\pi}_1^{c_1} \hat{\pi}_2^{c_2} \cdots \hat{\pi}_k^{c_k}$. If x is a valid double-descent/double-ascent of $|\pi|$, then the pack of letters $\hat{\pi}_x^{c_x}$ is called a *(valid) double-descent/double-ascent pack* of π. We call two elements in $\mathfrak{S}'_{\mathbf{m},i}$ *equivalent* if one can be obtained from the other by a sequence of actions of the form $\pi \mapsto \bar{\varphi}'_x(\pi)$. This defines an equivalence relation on $\mathfrak{S}_{\mathbf{m},i}$, since a pack $\hat{\pi}_x^{c_x}$ is a double-descent pack of π if and only if it is a double-ascent pack of $\bar{\varphi}'_x(\pi)$. Moreover, the equivalence class containing π, denoted $\mathrm{Orb}(\pi)$, has a unique permutation $\tilde{\pi}$ such that $\mathsf{vdd}(\tilde{\pi}) = 0$, the one with least descents. Therefore,

$$\sum_{\sigma \in \mathrm{Orb}(\pi)} x^{\mathsf{des}(\sigma)-1} = x^{\mathsf{des}(\tilde{\pi})-1}(1+x)^{\mathsf{vda}(\tilde{\pi})} = x^{\mathsf{des}(\tilde{\pi})-1}(1+x)^{m-i-2(\mathsf{des}(\tilde{\pi})-1)}, \tag{6.2}$$

where the second equality follows from the relationship

$$\mathsf{vda}(\tilde{\pi}) = m - i - 2(\mathsf{des}(\tilde{\pi}) - 1). \tag{6.3}$$

By the increasing factorization of multipermutations, it suffices to prove (6.3) for $\tilde{\pi}$ being a dominated word, i.e., a word with only one record. We can further assume that $\tilde{\pi}$ has no plateau and $m > 1$. For such a $\tilde{\pi}$, Eq. (6.3) reduces to

$$\mathsf{vda}(\tilde{\pi}) = m - 2(\mathsf{des}(\tilde{\pi}) - 1).$$

This is a consequence of the following simple fact

$$1+\mathsf{idd}(\tilde\pi)+\overline{\mathsf{peak}}(\tilde\pi)+\mathsf{ida}(\tilde\pi)+\overline{\mathsf{val}}(\tilde\pi)+\mathsf{vda}(\tilde\pi)=m,$$
$$1+\mathsf{idd}(\tilde\pi)+\overline{\mathsf{peak}}(\tilde\pi)=\mathsf{ida}(\tilde\pi)+\overline{\mathsf{val}}(\tilde\pi)=\mathsf{des}(\tilde\pi)-1,$$

where $\mathsf{idd}(\tilde\pi)/\mathsf{ida}(\tilde\pi)$ is the number of invalid double-descents/double-ascents of $\tilde\pi$ and $\overline{\mathsf{peak}}(\tilde\pi)$ (resp. $\overline{\mathsf{val}}(\tilde\pi)$) is the number of indices $1\le i\le m$ such that $\tilde\pi_{i-1}<\tilde\pi_i>\tilde\pi_{i+1}$ (resp. $\tilde\pi_{i-1}>\tilde\pi_i<\tilde\pi_{i+1}$) with the convention that $\tilde\pi_0=\tilde\pi_{m+1}=+\infty$.

Let $\widetilde{\mathfrak{S}'_{\mathbf{m},i}}:=\{\tilde\pi\in\mathfrak{S}'_{\mathbf{m},i}:\mathsf{vdd}(\tilde\pi)=0\}$. Using (6.2) and summing over all equivalence classes of $\mathfrak{S}'_{\mathbf{m},i}$ yields

$$\sum_{\pi\in\mathfrak{S}'_{\mathbf{m},i}}x^{\mathsf{des}(\pi)-1}=\sum_{\tilde\pi\in\widetilde{\mathfrak{S}'_{\mathbf{m},i}}}x^{\mathsf{des}(\tilde\pi)-1}(1+x)^{m-i-2(\mathsf{des}(\tilde\pi)-1)}.$$

It then follows that

$$\begin{aligned}y^{m-i}\sum_{\pi\in\mathfrak{S}'_{\mathbf{m},i}}(xy^{-1})^{\mathsf{des}(\pi)-1}&=y^{m-i}\sum_{\tilde\pi\in\widetilde{\mathfrak{S}'_{\mathbf{m},i}}}(xy^{-1})^{\mathsf{des}(\tilde\pi)-1}(1+xy^{-1})^{m-i-2(\mathsf{des}(\tilde\pi)-1)}\\&=\sum_{\tilde\pi\in\widetilde{\mathfrak{S}'_{\mathbf{m},i}}}(xy)^{\mathsf{des}(\tilde\pi)-1}(x+y)^{m-i-2(\mathsf{des}(\tilde\pi)-1)},\end{aligned}$$

which is equivalent to (1.6) in view of (6.1). □

Let $[2,n]:=\{2,3,\dots,n\}$. For $j\ge0$, introduce

$$D_{\mathbf{m},j}:=\{\pi\in\mathfrak{S}_m:\mathrm{Wasc}(\pi)\in\mathrm{Stab}([2,n])\text{ and }|\mathrm{Wasc}(\pi)|=j\}.$$

Theorem 6.2 (Linusson, Shareshian and Wachs)**.** *For any $\mathbf{m}\in\mathbb{P}^n$ and $j\ge0$,*

$$\bar\gamma_{\mathbf{m},0,j}=|D_{\mathbf{m},j+1}|.$$

Proof. Let

$$\overline{\Gamma}_{\mathbf{m},j}:=\{\pi\in\mathfrak{S}_{\mathbf{m}}:\mathsf{vdd}(\pi)=\mathsf{hfix}(\pi)=0,\mathsf{des}(\pi)=j\}.$$

An action $\pi\mapsto\bar\varphi_x(\pi)$ is called an *invalid action* on π if x is an invalid double descent of π. Applying all possible invalid actions to an element $\pi\in\overline{\Gamma}_{\mathbf{m},j}$ results in a permutation $g(\pi)\in D_{\mathbf{m},j}$. It is routine to check that the mapping $\pi\mapsto g(\pi)$ is a bijection between $\overline{\Gamma}_{\mathbf{m},j}$ and $D_{\mathbf{m},j}$. Thus, $|\overline{\Gamma}_{\mathbf{m},j}|=|D_{\mathbf{m},j}|$ and so the result follows from Theorem 1.6. □

7. Concluding remarks, open problems

The Foata–Strehl group action on permutations was invented by Foata and Strehl [21] (see also [39]) to prove combinatorially the homogeneous γ-positivity expansion

$$A_n(x,y)=\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}\gamma_{n,k}(xy)^k(x+y)^{n+1-2k},$$

where $\gamma_{n,k}$ enumerates the permutations in $\mathfrak{S}_n$ with k descents and with no double descents. Since then some generalizations and analogues of this γ-positivity expansion, with or without combinatorial proofs, have been found [9, 22, 29, 30, 32, 38]. The reader is referred to

Athanasiadis's survey [4] for the state-of-the-art on this theme. In the following, we discuss some recent developments and open problems along this line of research.

On the one hand, various interesting applications and extensions of the Foata–Strehl actions have been found in the literature:

- Brändén [9] successfully applied the *Modified Foata–Strehl action* (*MFS-action* for short) to prove the γ-positivity of the descent polynomials on r-stack sortable permutations. In the same paper, he also extended the MFS-action to linear extensions of sign-graded posets to give a new proof of the unimodality of the (P,ω)-Eulerian polynomials of sign-graded posets.
- Postnikov–Reiner–Williams [38] developed the MFS-action to prove the γ-positivity of the h-polynomials of various families of graph-associahedra.
- Lin and Zeng [32] applied a restricted version of the MFS-action to prove the q-γ-positivity of the basic Eulerian polynomials.
- Lin and Kim [29] applied the MFS-action to $(2413, 4213)$-avoiding permutations and 021-avoiding inversion sequences.
- Athanasiadis [4] generalized the MFS-action to Smirnov words, which was further generalized to all words in the proof of Theorem 1.4 in Section 5.
- Very recently, Lin–Ma–Ma–Zhou [31] successfully generalized the MFS-action from increasing trees to weakly increasing trees of a multiset.

On the other hand, several interesting generalizations of the Eulerian polynomials were proved to have nonnegative γ-coefficients, but to find a combinatorial interpretation of the corresponding γ-coefficients is widely open. Three representative examples are:

- The *symmetric restricted Eulerian polynomials* have γ-positivity expansion [37]:
$$\sum_{\substack{\pi\in\mathfrak{S}_n\\ \pi_1=j,n+1-j}} t^{\mathsf{des}(\pi)}=\sum_{i=1}^{\lfloor (n+1)/2\rfloor}\gamma_i^{(n,j)}t^i(1+t)^{n+1-2i},$$
where $\gamma_i^{(n,j)}$ are nonnegative integers.
- The *double Eulerian polynomials* have two-sided γ-positivity expansion [30]:
$$\sum_{\pi\in\mathfrak{S}_n} s^{\mathsf{des}(\pi^{-1})}t^{\mathsf{des}(\pi)}=\sum_{\substack{i\geq 1,j\geq 0\\ j+2i\leq n+1}} a_{n,i,j}(st)^i(1+st)^j(s+t)^{n+1-j-2i},$$
where $a_{n,i,j}$ are nonnegative integers.
- The *k-multiset Eulerian polynomials* (for $k\geq 1$)
$$A_n^{(k)}(t):=\sum_{\pi\in\mathfrak{S}_{\mathbf{m}}} t^{\mathsf{des}(\pi)}\quad\text{with } \mathbf{m}=(k,k,\dots,k)\in\mathbb{P}^n$$
are known to be palindromic [13] and real-rooted [41], and thus are γ-positive.

Note that the symmetric restricted Eulerian polynomials are the h-polynomials of the barycentric subdivision of a boolean complex (see [37]), while the 2-multiset Eulerian polynomials $A_n^{(2)}(t)$ are proved only recently by Ardila [2] to be the h-polynomial of the bipermutahedral fan. It would be interesting to see whether there are geometry meanings for the k-multiset Eulerian polynomials $A_n^{(k)}(t)$ (for $k \geq 3$) and for the double Eulerian polynomials. It should be pointed out that an interpretation in terms of some kind of weakly increasing trees for the γ-coefficients of $A_n^{(2)}(t)$ were found very recently in [31]. It seems hard to develop group action proofs of the above three γ-positivity generalized Eulerian polynomials.

In this paper, we prove three instances of partial γ-positivity polynomials that enumerate classical statistics on multipermutations via developing generalizations of the MFS-action. No geometry interpretation for these three classes of generalized Eulerian polynomials were known in general, except for the classical Eulerian polynomials. To end this paper, we pose a partial γ-positivity conjecture related to enumerative polynomials on quai-Stirling permutations introduced by Archer, Gregory, Pennington and Slayden [1].

A word $W = w_1 w_2 \cdots w_n$ is said to avoid the word (or pattern) $P = p_1 p_2 \cdots p_k$ ($k \leq n$) if there does not exist $i_1 < i_2 < \cdots < i_k$ such that the subword $w_{i_1} w_{i_2} \cdots w_{i_k}$ of W is order isomorphic to P. In the language of pattern avoidance, a multipermutation is *Stirling* if and only if it avoids the pattern 212. Analogously, a multipermutation is *quasi-Stirling* if and only if it avoids both the patterns 1212 and 2121. For any $\mathbf{m} \in \mathbb{P}^n$, let $\overline{\mathcal{Q}}_{\mathbf{m}}$ denote the set of all quasi-Stirling permutations in $\mathfrak{S}_{\mathbf{m}}$. The set $\overline{\mathcal{Q}}_{\mathbf{m}}$ when $\mathbf{m} = (2, 2, \ldots, 2)$ was first considered and enumerated by Archer et al. [1] and further generalized to $\mathbf{m} = (k, k, \ldots, k)$ (for $k \geq 1$) by Elizalde in [19], where the joint distribution of the three statistics asc, plat and des over these special quasi-Stirling permutations was also investigated. Note that $\mathcal{Q}_{\mathbf{m}} \subseteq \overline{\mathcal{Q}}_{\mathbf{m}} \subseteq \mathfrak{S}_m$ and in view of Theorems 1.4 and 1.2 and Corollary 5.2, we make the following conjecture basing on empirical evidence.

Conjecture 7.1. *For any $\mathbf{m} \in \mathbb{P}^n$, introduce*

$$Q_{\mathbf{m}}(x, y, z) := \sum_{\pi \in \overline{\mathcal{Q}}_{\mathbf{m}}} x^{\mathsf{asc}(\pi)} y^{\mathsf{des}(\pi)} z^{\mathsf{plat}(\pi)}.$$

Then, $Q_{\mathbf{m}}(x, y, z)$ is partial γ-positive.

Lemma 5.1 is equivalent to the assertion that the modified WFS-action φ'_x preserves the pattern 212. However, this modified WFS-action φ'_x does not preserve the patterns 1212 or 2121; consider for instance the action φ'_1 or φ'_4 on 23123.

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(Zhicong Lin) Research Center for Mathematics and Interdisciplinary Sciences, Shandong University, Qingdao 266237, P.R. China
Email address: linz@sdu.edu.cn

(Jun Ma) School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China
Email address: majun904@sjtu.edu.cn

(Philip B. Zhang) College of Mathematical Science, Tianjin Normal University, Tianjin 300387, P.R. China
Email address: zhang@tjnu.edu.cn