Multivariate stable Eulerian polynomials on segmented permutations

Philip B. Zhang[1] and Xutong Zhang[2]

College of Mathematical Science
Tianjin Normal University, Tianjin 300387, P. R. China

Email: [1]zhang@tjnu.edu.cn   [2]zhang.xutong@foxmail.com

Abstract. Recently, Nunge studied Eulerian polynomials on segmented permutations, namely *generalized Eulerian polynomials*, and further asked whether their coefficients form unimodal sequences. In this paper, we prove the stability of the generalized Eulerian polynomials and hence confirm Nunge's conjecture. Our proof is based on Brändén's stable multivariate Eulerian polynomials. By acting on Brändén's polynomials with a stability-preserving linear operator, we get a multivariate refinement of the generalized Eulerian polynomials. To prove Nunge's conjecture, we also develop a general approach to obtain generalized Sturm sequences from bivariate stable polynomials.

1   Introduction

The main objective of this paper is to prove a unimodality conjecture for the generalized Eulerian polynomials proposed by Nunge. In this paper, we give a multivariate refinement of the generalized Eulerian polynomials and then prove their stability, from which Nunge's conjecture can be confirmed.

Let $\mathfrak{S}_n$ denote the set of permutations of $[n] := \{1, 2, \ldots, n\}$. Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathfrak{S}_n$, let

$$\mathrm{des}(\pi) = |\{i \in [n-1] : \pi_i > \pi_{i+1}\}|$$

denote the descent number of π. The Eulerian number $A(n, k)$ is defined as the number of permutations with k descents in $\mathfrak{S}_n$ and the Eulerian polynomial $A_n(t)$ is usually defined as the ordinary generating function of $A(n, k)$, namely,

$$A_n(t) = \sum_{k=0}^{n-1} A(n, k) t^k = \sum_{\pi \in \mathfrak{S}_n} t^{\mathrm{des}(\pi)}.$$

Eulerian polynomials and Eulerian numbers have been extensively studied over the years, see [6, 12].

Corteel and Nunge [5] studied a recoil statistic on partially signed permutations during their study of the 2-species exclusion processes and Hopf algebras over segmented compositions, see also [11]. Recently, Nunge [10] gave a corresponding statistic on segmented permutations, namely the descent statistic. A *segmented permutation* is a permutation equipped with some separate bars which can be inserted into the slots between two consecutive letters. Let $\mathfrak{P}_n$ denote the set of segmented permutations of $[n]$. For $\sigma \in \mathfrak{P}_n$, a position i is said to be a *descent* if $\sigma_i > \sigma_{i+1}$ and there is no bar in the slot between σ_i and σ_{i+1}. We denote by $\mathrm{des}(\sigma)$ (respectively, $\mathrm{seg}(\sigma)$) the number of descents (respectively, bars) of σ. For example, with $\sigma = 2|516|34$, we have $\mathrm{des}(\sigma) = 1$ and $\mathrm{seg}(\sigma) = 2$. The generalized Eulerian numbers are defined as follows:

$$T(n, k) = |\{\sigma \in \mathfrak{P}_n : \mathrm{des}(\sigma) = k\}|$$

and

$$K(n, i, j) = |\{\sigma \in \mathfrak{P}_n : \mathrm{des}(\sigma) = i, \ \mathrm{seg}(\sigma) = j\}|.$$

Following Nunge, let

$$P_n(t) = \sum_{k=0}^{n-1} T(n, k) t^k = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)}$$

and

$$\alpha_n(t, q) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-i-1} K(n, i, j) t^i q^j = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}.$$

Note that the polynomial $\alpha_n(t, q)$ gives back the usual Eulerian polynomials at $q = 0$ and the ordered Bell polynomials at $t = 0$. Namely, $\alpha_n(t, 0) = A_n(t)$ and $\alpha_n(0, q) = \sum_{r=0}^{n-1} (r + 1)! S(n, r + 1) q^r$, where $S(n, k)$ are the Stirling numbers of the second kind.

Nunge's conjecture is concerned with the unimodality of the rows and columns of $T(n, k)$ and $K(n, i, j)$. Recall that a sequence of positive integers $a_0, a_1, \ldots, a_n$ is said to be *unimodal* if there exists an index $0 \leq i \leq n$ such that $a_0 \leq a_1 \leq \cdots \leq a_{i-1} \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$. It is well known that, for a sequence of positive numbers, its log-concavity implies unimodality, see [14]. Nunge proposed the following conjecture.

Conjecture 1.1. *For any positive integer n, the integer sequences $\{T(n, k)\}_{k=0}^{n-1}$, $\{K(n, i, j)\}_{i=0}^{n-j-1}$ and $\{K(n, i, j)\}_{j=0}^{n-i-1}$ are unimodal sequences.*

By the Newton's inequality (see [8, p. 104]), if a polynomial with nonnegative coefficients has only real zeros, then the sequence of its coefficients is log-concave and hence unimodal. Given a positive integer n, for any $0 \leq i, j \leq n - 1$, let

$$K_{n,j}(x) = \sum_{i=0}^{n-1-j} K(n, i, j) x^i \quad \text{and} \quad L_{n,i}(x) = \sum_{j=0}^{n-1-i} K(n, i, j) x^j.$$

2

In this paper, we obtain the real-rootedness of $P_n(t)$, $K_{n,j}(x)$, and $L_{n,i}(x)$ from which the log-concavity and unimodality of their coefficients can be deduced and hence give an affirmative answer to Conjecture 1.1.

Theorem 1.2. *For any integers $n \geq 1$ and $0 \leq i, j \leq n - 1$, the polynomials $P_n(t)$, $K_{n,j}(x)$ and $L_{n,i}(x)$ have only real zeros.*

Our approach to prove Theorem 1.2 is to employ a multivariate stable polynomial, which generalizes both $P_n(t)$ and $\alpha_n(t, q)$. The theory of stable polynomials has turned out to play a key role in various combinatorial problems, see [2, 16] and references therein. In this paper, we introduce a new multivariate polynomial $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ and show that it can be constructed by acting on Brändén's multivariate stable Eulerian polynomial with a linear operator. We then prove this operator preserves stability and hence the polynomial $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ is stable.

Our next step is to deduce the real-rootedness of $P_n(t)$, $K_{n,j}(x)$, and $L_{n,i}(x)$ from the stability of $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$. We note that $\alpha_n(t, q)$ can be reduced from $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ and $P_n(t) = \alpha_n(t, 1)$. Since the involved operators preserve real stability, we obtain the stability of $\alpha_n(t, q)$ and $P_n(t)$. Hence, the univariate stable polynomial $P_n(t)$ has only real zeros. To prove the real-rootedness of $K_{n,j}(x)$ and $L_{n,i}(x)$, we develop a general way (Theorem 2.3) to obtain generalized Sturm sequences (defined in Section 2) from bivariate stable polynomials.

The remainder of this paper is organized as follows. In Section 2, we recall some definitions and results on stable polynomials, including certain linear operators which preserve real stability. We also give a new result (Theorem 2.3) which relates bivariate stable polynomials to generalized Sturm sequences. Section 3 is dedicated to our proof of Theorem 1.2. Our proof is based on the stability of $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$. We also relate $\alpha_n(t, q)$ and $P_n(t)$ to the classical Eulerian polynomial $A_n(t)$ and then prove the stability of $\alpha_n(t, q)$ in an alternative way.

2 Preliminaries

In this section, we shall give an overview of stable polynomials. After recalling the definition of stable polynomials, we list some stability-preserving linear operators which will be used in the paper. We also present Theorem 2.3, a general approach to obtain generalized Sturm sequences from bivariate stable polynomials. Our proof of Theorem 2.3 is similar to that of Newton's inequality and based on the Hermite–Biehler Theorem.

Now let us recall the notion of real stability, which generalizes the notion of real-rootedness from univariate real polynomials to multivariate real polynomials. For a positive integer n, let $\boldsymbol{x}$ be the n-tuple $(x_1, \ldots, x_n)$. Let $\mathbb{H}_+ = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ denote the open upper complex half-plane. A polynomial $f \in \mathbb{R}[\boldsymbol{x}]$ is said to be (real) *stable* if $f(\boldsymbol{x}) \neq 0$ for any $\boldsymbol{x} \in \mathbb{H}_+^n$ or f is identically zero. Note that a univariate polynomial $f(x) \in \mathbb{R}[x]$ is stable if and only if it has real zeros. A polynomial $f(\boldsymbol{x})$ is said to be *multiaffine* if the power of each indeterminate x_i is at most one. For a set $\mathcal{P}$ of polynomials, let $\mathcal{P}^{MA}$ be the set of multiaffine

polynomials in $\mathcal{P}$. Borcea and Brändén [1] gave a complete characterization of the linear operators which preserve multivariate stable polynomials. In this paper, we shall use a multiaffine version of Borcea and Brändén's characterization, which shall play a key role in this paper to prove the stability of polynomials.

Lemma 2.1 ([16, Theorem 3.5]). *Let* $T : \mathbb{R}[\boldsymbol{x}]^{MA} \to \mathbb{R}[\boldsymbol{x}]$ *be a linear operator acting on the variables* $\boldsymbol{x} = (x_1, \ldots, x_n)$. *If the polynomial*

$$T\left(\prod_{i=1}^{n}(x_i + \hat{x}_i)\right)$$

is a stable polynomial of variables $\boldsymbol{x}$ *and* $\hat{\boldsymbol{x}} = (\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n)$, *then* T *preserves real stability.*

Once multivariate polynomials are shown be stable, we can then reduce them to real stable univariate polynomials by using the following operations.

Lemma 2.2 ([16, Lemma 2.4]). *Given* $i, j \in [n]$, *the following operations preserve real stability of* $f \in \mathbb{R}[\boldsymbol{x}]$:

1. Differentiation: $f \mapsto \partial f / \partial x_i$.

2. Diagonalization: $f \mapsto f|_{x_i=x_j}$.

3. Specialization: *for* $a \in \mathbb{R}$, $f \mapsto f|_{x_i=a}$.

Given two real-rooted polynomials $f(x)$ and $g(x)$ with positive leading coefficients, we say that $g(x)$ *interlaces* $f(x)$, denoted $g(x) \preceq f(x)$, if

$$\cdots \leq s_2 \leq r_2 \leq s_1 \leq r_1,$$

where $\{r_j\}$ and $\{s_k\}$ are the sets of zeros of $f(x)$ and $g(x)$, respectively. Following Liu and Wang [9], a sequence $\{f_n(x)\}_{n \geq 0}$ of real polynomials with positive leading coefficients is said to be *a generalized Sturm sequence* if the polynomial $f_n(x)$ has only real roots and $f_n(x) \preceq f_{n+1}(x)$ for any $n \geq 0$.

The following theorem provides a general approach to obtain generalized Sturm sequences from bivariate stable polynomials.

Theorem 2.3. *Suppose that* $F(x, y) = \sum_{j=0}^{n} f_j(x)y^j$ *is a bivariate polynomial with real coefficients. If* $F(x, y)$ *is stable, then the polynomial* $f_j(x)$ *has only real zeros for any* $0 \leq j \leq n$ *and moreover* $\{f_{n-j}(x)\}_{j=0}^{n}$ *forms a generalized Sturm sequence.*

In order to prove Theorem 2.3, we need the Hermite–Biehler Theorem, which reveals the close connection between interlacing and stability.

Lemma 2.4 (Hermite–Biehler, [13, Th. 6.3.4]). *Let* $f(z)$ *and* $g(z)$ *be two non-constant polynomials with real coefficients. Then the following statements are equivalent:*

4

- $f(z)$ and $g(z)$ have only real zeros, and moreover $g(z) \preceq f(z)$;

- the polynomial $f(z) + ig(z)$ is stable.

Now we are at the position to give a proof of Theorem 2.3. Note that our proof is similar to that of Newton's inequality, see [8, p. 104].

Proof of Theorem 2.3. By taking the k-th order partial derivative with respect to y of the real stable polynomial

$$F(x, y) = \sum_{j=0}^{n} f_j(x) y^j,$$

it follows from Lemma 2.2 that

$$\sum_{j=k}^{n} (j)_k f_j(x) y^{j-k}$$

is real stable, where $(j)_k = j(j-1)\cdots(j-k+1)$. Note that if $y \in \mathbb{H}_+$ then $-\frac{1}{y} \in \mathbb{H}_+$. Hence, we obtain the real stability of

$$y^{n-k} \sum_{j=k}^{n} (j)_k f_j(x) (-\frac{1}{y})^{j-k} = \sum_{j=k}^{n} (-1)^{j-k} (j)_k f_j(x) y^{n-j}. \tag{1}$$

Similarly, by taking the $(n-k-1)$-th order the partial derivative with respect to y of (1), we get the real stability of

$$\sum_{j=k}^{k+1} (-1)^{j-k} (j)_k (n-j)_{n-k-1} f_j(x) y^{k-j+1}$$

$$= k!(n-k)! f_k(x) y - (k+1)!(n-k-1)! f_{k+1}(x)$$

$$= (k+1)!(n-k-1)! \left(\frac{n-k}{k+1} f_k(x) y - f_{k+1}(x) \right).$$

Replacing y by $\dfrac{k+1}{n-k} y$, it follows that

$$f_{k+1}(x) - f_k(x) y$$

is real stable and so is

$$y(f_{k+1}(x) - f_k(x)(-\frac{1}{y})) = f_k(x) + y f_{k+1}(x).$$

Therefore, it follows from Lemma 2.4 that $f_{k+1}(x) \preceq f_k(x)$. This completes the proof of Theorem 2.3. □

3   Proof of Theorem 1.2

The main objective of this section is to prove Theorem 1.2. We first recall Brändén's multivariate Eulerian polynomial and then introduce a multivariate refinement of Nunge's Eulerian polynomials on segmented permutations. We next show that it can be obtained by acting on Brändén's multivariate Eulerian polynomial with a stability-preserving linear operator.

Before presenting our result, let us first recall Brändén's multivariate Eulerian polynomial. Given a permutation $\pi \in \mathfrak{S}_n$, let

$$\mathcal{DT}(\pi) = \{\pi_i : \pi_i > \pi_{i+1}\} \quad \text{and}$$

$$\mathcal{AT}(\pi) = \{\pi_{i+1} : \pi_i < \pi_{i+1}\}$$

be the *descent top* set and the *ascent top* set, respectively. For $\mathcal{T}$ a set with entries from $[n]$, we let $\boldsymbol{x}^{\mathcal{T}} = \prod_{i \in \mathcal{T}} x_i$. Brändén defined a real multivariate polynomial $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$ as follows:

$$\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\pi \in \mathfrak{S}_n} w(\pi), \text{ where } w(\pi) = \boldsymbol{x}^{\mathcal{DT}(\pi)} \boldsymbol{y}^{\mathcal{AT}(\pi)}. \tag{2}$$

For example, $w(251634) = y_5 x_5 y_6 x_6 y_4$. Clearly, $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$ is multiaffine. Brändén proved the following result.

Lemma 3.1. *For any positive integer n, the polynomial $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$ is stable.*

For a proof of Lemma 3.1 and further generalizations, we refer the reader to [3, 4, 7, 15]. By diagonalizing the variables x_i to x, specializing y_i to 1, it follows that

$$A_n(x) = \sum_{\pi \in \mathfrak{S}_n} x^{|\mathcal{DT}(\pi)|} = \sum_{\pi \in \mathfrak{S}_n} x^{\text{des}(\pi)}$$

is stable. Since $A_n(x)$ is univariate, it is equivalent to say that $A_n(x)$ has only real zeros.

We next give our multivariate Eulerian polynomials on segmented permutations. Given a segmented permutation $\sigma \in \mathfrak{P}_n$, let

$\mathcal{DT}(\sigma) = \{\sigma_i : \sigma_i > \sigma_{i+1} \text{ and there is no bar in the slot between } \sigma_i \text{ and } \sigma_{i+1}\},$

$\mathcal{AT}(\sigma) = \{\sigma_{i+1} : \sigma_i < \sigma_{i+1} \text{ and there is no bar in the slot between } \sigma_i \text{ and } \sigma_{i+1}\},$

$\mathcal{DTS}(\sigma) = \{\sigma_i : \sigma_i > \sigma_{i+1} \text{ and there is a bar in the slot between } \sigma_i \text{ and } \sigma_{i+1}\}, \text{ and}$

$\mathcal{ATS}(\sigma) = \{\sigma_{i+1} : \sigma_i < \sigma_{i+1} \text{ and there is a bar in the slot between } \sigma_i \text{ and } \sigma_{i+1}\}$

be the *descent top* set, the *ascent top* set, the *descent top segment* set and the *ascent top segment* set, respectively. Let $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ be a real multivariate multiaffine polynomial defined as

$$\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) = \sum_{\sigma \in \mathfrak{P}_n} w'(\sigma), \quad \text{where } w'(\sigma) = \boldsymbol{x}^{\mathcal{DT}(\sigma)} \boldsymbol{y}^{\mathcal{AT}(\sigma)} \boldsymbol{z}^{\mathcal{DTS}(\sigma)} \boldsymbol{w}^{\mathcal{ATS}(\sigma)}.$$

$$\tag{3}$$

For example, $w'(2|516|34) = w_5 x_5 y_6 z_6 y_4$. The main result of this section is stated as follows.

Theorem 3.2. *For any positive integer n, the polynomial $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ is stable.*

Proof. To prove this theorem, we first establish an identity, which relates the polynomial $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ to Brändén's polynomial $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$. From the definition of $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$, it is clear to see that for any permutation π there is a one-to-one correspondence between variables appearing in $w(\pi)$ and slots of two adjacent letters in π. Given a permutation $\pi \in \mathfrak{S}_n$, we can generate a segmented permutation $\sigma \in \mathfrak{P}_n$ by deciding whether to insert a bar in every slot between two adjacent letters. If a bar is inserted in a slot followed by a descent top (respectively, preceding an ascent top), namely i, then the descent top (respectively, the ascent top) will be replaced by a descent top segment (respectively, an ascent top segment) and hence the corresponding variable x_i (respectively, y_i) will be replaced by z_i (respectively, w_i). Then, for any segmented permutation σ there is a one-to-one correspondence between variables appearing in $w'(\sigma)$ and slots of two adjacent letters in σ. Hence, together with the fact that both $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ are multiaffine, we obtain that

$$\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}) = \prod_{j=2}^{n}(1 + w_j \frac{\partial}{\partial y_j})(1 + z_j \frac{\partial}{\partial x_j}) \boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y}). \tag{4}$$

We next prove the stability of $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ via (4). For $2 \le j \le n$, let $T_j = 1 + z_j \frac{\partial}{\partial x_j}$ be a linear operator defined on $\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}]^{MA}$. Since the polynomial

$$T_j \Big(\prod_{i=1}^{n}(x_i + \hat{x}_i)(y_i + \hat{y}_i)(z_i + \hat{z}_i)(w_i + \hat{w}_i) \Big)$$

$$= (x_j + \hat{x}_j + z_j) \prod_{k \ne j}(x_k + \hat{x}_k) \times \prod_{i=1}^{n}(y_i + \hat{y}_i)(z_i + \hat{z}_i)(w_i + \hat{w}_i)$$

is stable, it follows from Lemma 2.1 that the linear operator T_j preserves stability, Similarly, the linear operator $1 + w_j \frac{\partial}{\partial y_j}$ defined on $\mathbb{R}[\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w}]^{MA}$ also preserves stability for any $2 \le j \le n$. Therefore, we obtain that their product $\prod_{j=2}^{n}(1 + w_j \frac{\partial}{\partial y_j})(1 + z_j \frac{\partial}{\partial x_j})$ preserves stability. Since $\boldsymbol{A}_n(\boldsymbol{x}, \boldsymbol{y})$ is stable, we get the desired stable property of $\boldsymbol{\alpha}_n(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}, \boldsymbol{w})$ by (4). This completes the proof. $\square$

By diagonalizing the variables x_i to x, specializing y_i to 1, and substituting z_i and w_i by t, it follows that

Theorem 3.3. *For any positive integer n, the polynomial*

$$\alpha_n(t, q) = \sum_{\sigma \in \mathfrak{P}_n} t^{\mathrm{des}(\sigma)} q^{\mathrm{seg}(\sigma)}$$

is stable.

Now it is time for us to prove Theorem 1.2.

Proof of Theorem 1.2. Since $\alpha_n(t,1) = P_n(t)$, if follows that $P_n(t)$ is stable and hence real-rooted as a univariate polynomial with real coefficients. For the real-rootedness of $K_{n,j}(x)$ and $L_{n,i}(x)$, we shall apply Theorem 2.3 to $\alpha_n(t,q)$. Since

$$\alpha_n(t,q) = \sum_{j=0}^{n-1} K_{n,j}(t)q^j,$$

we get that the polynomial sequence $\{K_{n,n-j-1}(t)\}_{j=0}^{n-1}$ forms a generalized Sturm sequence. Similarly, since

$$\alpha_n(t,q) = \sum_{i=0}^{n-1} L_{n,i}(q)t^i,$$

we get that the polynomial sequence $\{L_{n,n-i-1}(t)\}_{i=0}^{n-1}$ forms a generalized Sturm sequence. This completes the proof of Theorem 1.2. $\qquad\square$

Before ending this paper, we would like to express $\alpha_n(t,q)$ and $P_n(t)$ in terms of $A_n(t)$, which leads to an alternative proof of Theorem 3.3.

Theorem 3.4. *For any positive integer n, we have*

$$\alpha_n(t,q) = (1+q)^{n-1} A_n\left(\frac{t+q}{1+q}\right), \tag{5}$$

and

$$P_n(t) = 2^{n-1} A_n\left(\frac{t+1}{2}\right). \tag{6}$$

Proof. By the following identity [10, Corollary 2.5]:

$$K(n,i,j) = \sum_{k=0}^{n-1} \binom{k}{i}\binom{n-1-k}{i+j-k} A(n,k),$$

we have that

$$\begin{aligned}
\alpha_n(t,q) &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-i-1}\sum_{k=0}^{n-1} \binom{k}{i}\binom{n-1-k}{i+j-k} A(n,k)t^i q^j \\
&= \sum_{k=0}^{n-1} A(n,k) \sum_{i=0}^{k} \binom{k}{i}t^i q^{k-i} \sum_{j=k-i}^{n-i-1}\binom{n-1-k}{i+j-k} q^{i+j-k} \\
&= \sum_{k=0}^{n-1} A(n,k) \sum_{i=0}^{k} \binom{k}{i}t^i q^{k-i}(1+q)^{n-1-k}
\end{aligned}$$

8

$$= \sum_{k=0}^{n-1} A(n,k)(1+q)^{n-1-k}q^k \sum_{i=0}^{k} \binom{k}{i}(t/q)^i$$

$$= \sum_{k=0}^{n-1} A(n,k)(1+q)^{n-1-k}q^k(1+t/q)^k$$

$$= \sum_{k=0}^{n-1} A(n,k)(1+q)^{n-1-k}(t+q)^k$$

$$= (1+q)^{n-1} \sum_{k=0}^{n-1} A(n,k)\left(\frac{t+q}{1+q}\right)^k.$$

This completes the proof of (5). Since $P_n(t) = \alpha_n(t,1)$, the equation (6) follows from (5). This completes the proof. □

Alternative proof of Theorem 3.3. Let $t \in \mathbb{H}_+$ and $q \in \mathbb{H}_+$. Then we have that $t+q \in \mathbb{H}_+$ and $1+q \in \mathbb{H}_+$ and thus $\frac{t+q}{1+q}$ will not be a negative real number. Since the Eulerian polynomial $A_n(x)$ has only negative real zeros, the polynomial $\alpha_n(t,q)$ will be non-zero whenever $t \in \mathbb{H}_+$ and $q \in \mathbb{H}_+$. Therefore, the polynomial $\alpha_n(t,q)$ is stable. □

We remark that comparing (6) with the well-known formula of Eulerian polynomials:

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{k=0}^{\infty}(k+1)^n t^k,$$

it follows that

$$\frac{P_n(t)}{(1-t)^{n+1}} = \sum_{k=1}^{\infty}(1+t)^{k-1}\frac{k^n}{2^{k+1}}$$

which has appeared in [10, Proposition 3.5].

References

[1] J. Borcea and P. Brändén, The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability, Invent. Math., 177 (2009), 541–569.

[2] P. Brändén, Unimodality, log-concavity, real-rootedness and beyond, in Handbook of Enumerative Combinatorics, Edited by Miklós Bóna, CRC Press, Boca Raton, FLress, 2015, 437–484.

[3] P. Brändén, J. Haglund, M. Visontai, and D. G. Wagner, Proof of the monotone column permanent conjecture, in Notions of positivity and the geometry of polynomials, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2011, 63–78.

[4] W. Y. Chen, R. X. Hao, and H. R. Yang, Context-free grammars and multivariate stable polynomials over stirling permutations, arXiv:1208.1420.

[5] S. Corteel and A. Nunge, 2-species exclusion processes and combinatorial algebras, Sém. Lothar. Combin., 78B (2017), Art. 56, 12.

[6] D. Foata, Eulerian polynomials: from Euler's time to the present, in The legacy of Alladi Ramakrishnan in the mathematical sciences, Springer, New York, 2010, 253–273.

[7] J. Haglund and M. Visontai, Stable multivariate Eulerian polynomials and generalized Stirling permutations, European J. Combin., 33 (2012), 477–487.

[8] G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, at the University Press, 1952. 2d ed.

[9] L. Liu, Y. Wang, A unified approach to polynomial sequences with only real zeros, Adv. in Appl. Math., 38 (2007), 542–560.

[10] A. Nunge, Eulerian polynomials on segmented permutations, ArXiv:1805.01797.

[11] A. Nunge, An equivalence of multistatistics on permutations, J. Combin. Theory Ser. A, 157 (2018), 435–460.

[12] T. K. Petersen, Eulerian numbers, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks], Birkhäuser/Springer, New York, 2015.

[13] Q. I. Rahman and G. Schmeisser, Analytic Theory of Polynomials, vol. 26 of London Mathematical Society Monographs. New Series, The Clarendon Press Oxford University Press, Oxford, 2002.

[14] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, in Graph theory and its applications: East and West (Jinan, 1986), vol. 576 of Ann. New York Acad. Sci., New York Acad. Sci., New York, 1989, 500–535.

[15] M. Visontai and N. Williams, Stable multivariate W-Eulerian polynomials, J. Combin. Theory Ser. A, 120 (2013), 1929–1945.

[16] D. G. Wagner, Multivariate stable polynomials: theory and applications, Bull. Amer. Math. Soc., 48 (2011), 53–84.